MARILYN BURNS is a nationally known mathematics educator whose messages about teaching math have reached teachers through her many Math Solutions Professional Development books, videotapes, audiocassettes, and extensive inservice programs. She is the recipient of numerous awards, including the Glenn Gilbert National Leadership Award, given by the National Council of Supervisors of Mathematics, and the Louise Hay Award for Contributions to Mathematics Education, given by the Association for Women in Mathematics.

50 PROBLEM-SOLVING LESSONS

Since 1986, Marilyn Burns has been writing and publishing the Math Solutions newsletter to offer support to teachers searching for new ways to teach mathematics. Each issue, which now can be found on our Web site at www.mathsolutions.com, is filled with articles grounded in the realities of the classroom. The articles present new ideas for classroom teaching, share new approaches to existing ideas, offer tips for classroom organization, and address general issues about math education.

In addition, each issue contains classroom activities, presented as vignettes, aimed at grades 1–6. Many lessons come from Marilyn Burns’s colleagues or from correspondence with teachers across the country who have attended Math Solutions Inservice courses. 50 Problem-Solving Lessons is a compilation of the best of these classroom-tested lessons. The lessons span the strands of the math curriculum and are illustrated with actual children’s work.

OTHER BOOKS FROM MATH SOLUTIONS PUBLICATIONS

A Message from Marilyn Burns

We at Math Solutions Professional Development believe that teaching math well calls for increasing our understanding of the math we teach, seeking deeper insights into how children learn mathematics, and refining our lessons to best promote students’ learning.

Math Solutions Publications shares classroom-tested lessons and teaching expertise from our faculty of Math Solutions Inservice instructors as well as from other respected math educators. Our publications are part of the nationwide effort we’ve made since 1984 that now includes

• more than five hundred face-to-face inservice programs each year for teachers and administrators in districts across the country;
• annually publishing professional development books, now totaling more than sixty titles and spanning the teaching of all math topics in kindergarten through grade 8;
• four series of videotapes for teachers, plus a videotape for parents, that show math lessons taught in actual classrooms;
• on-site visits to schools to help refine teaching strategies and assess student learning; and
• free online support, including grade-level lessons, book reviews, inservice information, and district feedback, all in our quarterly Math Solutions Online Newsletter.

For information about all of the products and services we have available, please visit our website at www.mathsolutions.com. You can also contact us to discuss math professional development needs by calling (800) 868-9092 or by sending an email to info@mathsolutions.com.

We’re always eager for your feedback and interested in learning about your particular needs. We look forward to hearing from you.
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When I was a new teacher, I was constantly searching for teaching ideas to use with my students. I combed every resource book I could find. I subscribed to teacher magazines. I talked with other teachers. I signed up for workshops whenever I could. As with all new teachers, my classroom experience was limited and my store of ideas was slim. Sunday nights typically wound up in a frenzied struggle to plan the week’s curriculum in ways that (I hoped) would be motivating for the students and manageable for me.

Now, 33 years later, my store of ideas is no longer slim, and I can rely on my years of experience. But I’m still searching for ideas to enhance and expand my teaching repertoire. While I’m no longer looking for ideas to fill an empty larder, I realize the benefit of refining and expanding my teaching by adding new ideas and approaches. And even with my years of experience, trying new lessons is still a challenge. Focusing on doing something new in the classroom calls for teaching and thinking at the same time, without the experience of being able to predict how students will respond. It’s this challenge that has always helped keep teaching alive and exciting for me.

In the summer of 1984, I began teaching Math Solutions five-day summer courses for teachers in kindergarten through grade 8 and established the Math Solutions faculty to offer the courses nationwide. Previously, I had been involved with several mathematics inservice projects and enjoyed helping teachers think in new ways about teaching mathematics. As the Math Solutions courses grew, I realized that teachers needed help beyond the five-day summer experience. We began presenting Math Solutions one-day workshops during the school year, both to offer follow-up support to teachers who had attended a summer course and to provide beginning experiences for teachers who were interested in thinking more about their math teaching.

In the spring of 1986, in order to provide another way to offer support to teachers searching for new ways to teach mathematics, I wrote the first Math Solutions newsletter and mailed it to
all of the teachers who had attended Math Solutions courses and workshops. Since then, I’ve written one or two newsletters each year, and our mailing list has grown to more than 40,000 teachers. Over the years, I’ve tried to keep all of the articles grounded in the realities of the classroom by presenting new ideas for classroom teaching, sharing new approaches to existing ideas, offering tips for classroom organization, and addressing general issues about math education.

To write the newsletters, I depended on the teaching ideas and classroom experiences of other teachers. Many ideas came from colleagues I worked with on a regular basis; some came from correspondence I received from teachers who had attended courses. Some articles offered activities that were new to me; others recycled familiar ideas, giving them new twists and energy. I wrote the classroom activities as vignettes, including details about classroom management as well as information about the mathematics being presented. As often as possible, articles were illustrated with samples of actual student work.

Over the years, writing the newsletters sparked other projects. I used articles from newsletters for the bulk of the ideas in *Math and Literature (K-3)*. I expanded some articles and included them in the series of *A Collection of Math Lessons*, and some ideas found their way into *Math By All Means* units.

For this book, I combed all of the back issues of the Math Solutions newsletters and identified articles that hadn’t appeared in other publications or, if they did appear, had been much revised. I chose articles that presented practical, classroom-tested instructional ideas and compiled them into this resource of 50 lessons for teaching mathematics in grades 1 through 6. In some cases, I used my more recent experiences with the lessons to boost articles with additional instructional ideas or samples of children’s work.

On the first page of each lesson, I indicate the span of grades for which the lesson is appropriate and the mathematics strands it addresses. The charts on pages 4–7 provide an overview of the strands and recommended grade levels for all the lessons.

Revisiting the past newsletters reminded me that teaching never stays the same. Improving and refining ideas is an ongoing part of the craft of teaching, and I hope that this book helps teachers examine and expand their classroom repertoires.
# Grade Levels for Activities

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THE LESSONS
Bonnie Tank taught this lesson to first graders in San Francisco, California, early in the school year as she was getting to know the children. Bonnie used this lesson to learn about how the students would represent a situation mathematically.

**Materials**
- Color Tiles
- Lunch bags with fewer than 10 Color Tiles in each (tiles in varying combinations of three colors), one per pair of students

Typically, young children’s early experiences with mathematical recording come from completing problems on workbook pages. These pages often present isolated exercises that do not relate to children’s concrete experiences. They do not give children the opportunity to learn how to formulate and record their own ideas, and they do little to reveal to teachers how children think and what they understand.

This lesson had a different focus. To prepare for the lesson, Bonnie prepared 15 lunch bags, each with fewer than 10 Color Tiles in varying combinations of three colors. (For example, she put 7 red, 1 green, and 1 blue in one bag; 3 yellow, 3 green, and 2 red in another; and so on.)
Bonnie organized the students into pairs and gave each pair a bag of tiles and a sheet of paper. She then explained their task to them.

“Your job is to work with your partner and record on your paper exactly what’s in your bag,” she said. “You can use pictures, words, numbers—whatever makes sense to both of you so that someone else can tell from your paper what you have in the bag.”

The Children’s Work

The children used various ways to approach the problem and to record the contents of their bags. Alan and Maura, for example, divided the tiles between them. Maura took one of each color, Alan took the rest, and they each recorded what they had. Alan drew red squares to represent the six red tiles he had; Maura wrote 1 red, 1 blue, 1 green and drew one square of each color.

Alan drew six red tiles, and Maura drew one red, one blue, and one green tile.

Kate and Ryan also divided their paper in half. However, each drew a picture of all the tiles, producing two records. When they compared, Kate found that she had made an error by including an extra tile. She crossed it out. They agreed to write the number 9 to show how many tiles they had.

All of the students knew how many tiles they had and could report the colors when asked, though some were unable to record this information accurately, either pictorially or numerically. However, the activity engaged all of the children, and it helped Bonnie gain beginning insights into individuals’ thinking and ability to use mathematical symbols to represent their thinking.
This lesson draws on information about pets to provide students with a graphing experience and an opportunity to count and compare numbers. Carolyn Felux taught the lesson to first graders in San Antonio, Texas.

**Counting Cats**

**Materials**
- Interlocking (Multilink, Snap, or Unifix) cubes, one per student
- Two lunch bags, one labeled “Yes,” one labeled “No”

“I’m interested in learning who has a cat for a pet,” Carolyn said to the first graders. She quickly circulated through the class, giving each child one Snap Cube.

Then Carolyn showed the students the two lunch bags and asked them to read the label on each one. She then explained, “As I walk around the room, you’ll each put your cube in one of the bags. Put your cube in the Yes bag if you have a cat and in the No bag if you don’t. Then we’ll use the cubes to find out how many of you do and do not have cats.”

After Carolyn had collected the children’s cubes, she said, “First we’ll find out how many of you have cats. Count with me as I remove the cubes from the Yes bag.” As the students counted,
Carolyn snapped the cubes together to make a train. The train had seven cubes.

Before doing the same with the other bag, Carolyn asked, “How many students do we have in class today?” A buzz of conversation broke out in the room. A few children stood up to take a head count, and then several children raised their hands. Carolyn had several of them explain how they knew there were 21 children present.

Carolyn then asked, “Do you think there are more or fewer cubes in the No bag than in the Yes bag?” Again, a buzz of conversation broke out.

Carolyn called the class to attention and asked the children to raise their hands if they thought the No bag had more. Then she had them raise their hands if they thought the Yes bag had more. A few thought there were probably the same in each. Some weren’t sure and didn’t raise their hands at all.

“The Yes bag has seven cubes,” Carolyn reminded the class. “How many cubes do you think are in the No bag? Talk with your neighbor about this.”

After a minute or so, Carolyn called on volunteers to report their ideas. Some children reported a number and were able to explain how they figured. For example, Ali said, “We counted,” and demonstrated with her fingers.

Other children, however, were willing to give answers but couldn’t explain. Sally just shrugged when Carolyn asked for her reasoning, and Elliot said, “I guessed.”

Carolyn then removed the cubes from the No bag and snapped them into a train. The children counted along. This train was 14 cubes long. Standing the two trains together gave the children a concrete comparison.

Carolyn continued with more questions, all designed to probe the students’ thinking and help her assess their understanding.

- What can you tell me about these two stacks of cubes?
- How can we use the cubes to figure out how many children are in class altogether?
- How many more children do not have cats than do have cats?

- We can tell how many children have cats, but I don’t think we can tell how many pet cats we have from these cubes. Who can explain why I think that?
Bonnie Tank got the idea for this lesson from *Games for Math*, by Peggy Kaye, a book for parents about helping their K–3 children learn math. (See the Bibliography on page 179.) Bonnie taught the lesson to a first grade class in San Francisco, California.

Where’s the Penny?

**Materials**
- 10 small paper cups
- One penny

To prepare for the lesson, Bonnie labeled the cups “1st,” “2nd,” “3rd,” and so on, up to “10th.” She placed them upside down in order on the table at the front of the room and hid a penny under the seventh cup.

“Your job is to guess under which cup I hid the penny,” Bonnie told the children. “If you make a guess that isn’t right, I’ll give you a clue. I’ll tell you whether I put the penny under a cup before or after the one you chose.”

English was not the first language for more than half of the children. Not only was Bonnie interested in having the children become familiar
with ordinal numbers and learn to interpret clues logically, she was also interested in providing them with experiences hearing and speaking English. For that reason, she asked them to give their clues in complete sentences.

“When you guess,” she told them, “say ‘I think the penny is under the third cup.’ Or the sixth cup. Or whatever cup you’d like to guess.”

Gregory guessed first. “I think you put it in the first cup,” he said. Even though this wasn’t exactly the wording Bonnie had given, she accepted it because it was a complete sentence.

“The penny is under a cup that comes after the first cup,” Bonnie said.

Cynthia guessed next. “The tenth one,” she said.

“Can you say your guess in a whole sentence?” Bonnie asked “Start with ‘I think . . .’”

“I think the penny is under the tenth one,” Cynthia said.

“I put the penny under a cup before the tenth one,” Bonnie responded.

Marcie gave the next guess. “I think it’s under the fifth cup,” she guessed.

“I put the penny under a cup after the fifth one,” Bonnie answered.

“I think the penny is under the second cup,” Lawrence offered next. Neither Lawrence nor any of the other children seemed aware that this guess was redundant. Bonnie did not call this to the children’s attention, but responded as she had for the other guesses.

“I put the penny under a cup after the second one,” she said.

The guessing continued. Though the children guessed different cups each time, their guesses were random. Finally, on the eighth guess, Ryan gave the correct answer.

The children were eager to play again. To prepare, Bonnie blocked their view of the cups and moved the penny to another cup. For this round, the children again began by guessing the first and tenth cups. Still, none made use of the information from the clues. The game took six guesses.

Bonnie handled the third game differently. In response to the first three guesses, she told the class that the penny was in a cup after the first cup, before the tenth cup, and before the sixth cup. Then she asked the children a question.

“What do you now know from the clues?” she asked.

The next three children Bonnie called on did not answer her question but made new guesses. To each Bonnie gave the same response: “I’m not interested in a new guess now,” she said. “Instead, I’m interested in what the clues you already have tell you.”

Finally, Jessamy responded. “I don’t remember the first two clues,” she said, “but I remember the last clue. It comes before the sixth.”

“What does that tell you?” Bonnie probed.

“Oh yeah,” Jessamy continued, “it’s not under the first, so it’s under the second, third, fourth, or fifth.”

Some of the children agreed with Jessamy; others didn’t understand. Bonnie continued with more children’s guesses. With each clue, more children were able to explain what information they now had. It took the class four guesses in the third and fourth games.

Bonnie thought that it might have helped the children if after each guess she lifted the cup and turned it upright. However, the children’s strategies seemed to improve as they played more games, so this extra hint didn’t seem essential.

Over the next several days, the class played the game with children taking turns hiding the penny and giving clues.
Marge Genolio, a first grade teacher in San Francisco, California, gives her students regular experiences with estimating and comparing numbers. In this lesson, she had the children count the number of pockets on their clothing. The lesson gave students experience that helped build their understanding of large numbers and place value.

Materials

- Interlocking (Multilink, Snap, or Unifix) cubes

“How many pockets do you think we are wearing today altogether?” Marge asked her class one Monday morning. After giving the children a chance to share their estimates, Marge organized a way for them to find out.

She put a supply of Unifix cubes at each table and directed the children to put one cube in each of their pockets. Once they had done this, Marge removed the remaining cubes.

She directed the students to remove the cubes from their pockets, snap them into trains, and compare their trains with those of the other students at their table. Then Marge called the class to attention.

“Raise your hand if you have a train that’s the same length as someone else’s at your table,” she said. Marge had a few children show their trains, report how many cubes they had, and tell who had a train the same length.
“Now raise your hand if your train is longer than someone else’s train,” Marge said. After a few children reported, she said, “Raise your hand if your train is shorter than someone else’s train.” Again, she had several students report.

Marge gave further directions. “Now let’s find out how many cubes we have altogether. This will tell us how many pockets we have on our clothing today. I’d like you to combine the cubes at your table by snapping them into trains of 10.”

Marge collected the trains of 10 from each table and then combined the extra cubes from each group to make additional 10s. The class counted the 10s and extras and found that there were 68 pockets. Marge recorded on the board:

6 tens and 8 ones = 68

Although it’s obvious to adults that the digits in 68 represent how many 10s and 1s there are, this is not obvious to children. Experiences like these can help them make this connection.

Marge posted a sheet of paper, then wrote the date and recorded the number 68 on it. She told the children that they would try the activity again the next day.

“I wonder if we’ll get the same results tomorrow,” she mused. The children’s reactions were mixed.

The count rose on Tuesday and Wednesday, and by Thursday there were 95 pockets. On Friday, however, there was a slump—only 89 pockets. Some children were disappointed; they were hoping for 100. Obviously, the children had exhausted the pocket potential of their wardrobes by that time!

A Literature Connection

When Stephanie Sheffield did this lesson with her first graders, she first read aloud Peter’s Pockets by Eve Rice. The story is about a boy who puts on a new pair of pants and takes a walk with his uncle. As he finds treasures he wants to keep, Peter discovers that his pants have a serious flaw—no pockets! When Peter gets home, his mother solves the problem by sewing enough pockets on Peter’s pants to hold all of his treasures. You can read Stephanie’s version of the lesson in her book Math and Literature (K–3), Book Two. (See the Bibliography on page 179.)
Planting Bulbs

Each December, Carole Clarin’s K–1 students in New York City plant narcissus bulbs. For several weeks, children measure the growth of their bulbs, graph data, record findings in individual books, and contribute to a class timeline. This month-long project integrates reading, writing, math, science, and art, and results in a room full of delicious smelling flowers in time for the holidays.

**Materials**
- Narcissus bulbs, one per student (plus a few extra in case some students’ bulbs don’t grow)
- Toothpicks, one box
- Clear plastic cups, one per bulb
- Centicubes
- Two sheets of chart paper
- Circle stickers (available at stationery stores)
- Potting soil

To begin this project each year, Carole has each child select a bulb, support it with three toothpicks, and put it into water in a plastic cup. The children label their cups with their names. (Carole always plants several extra bulbs in case some don’t grow.)

Carole creates a class chart. She lists all of the students’ names, rules four columns, and titles the columns “Roots,” “Shoots,” “Buds,” and “Flowers.” Each child posts a circle sticker in the appropriate column when he or she first sees roots, shoots, buds, or flowers.
On one of the class charts, each child posts a circle sticker when he or she notices roots, shoots, buds, or flowers.

When all the bulbs have sprouted, which takes about a week, the children use Centicubes to measure the height of their shoots. Carole posts another chart on which the students record the lengths of their shoots by coloring in columns of centimeter squares.

A week later, each child takes a new measurement and adds the information to the graph, coloring his or her column so it represents the current height in Centicubes. Carole chooses a different color crayon for each week so that the children can see how much their bulbs grow each week.

As roots begin to grow, the students plant their bulbs in gravel or soil and continue to measure and record their data each week.

There is a great deal of discussion about the data on the graph. Carole asks questions such as:

- What does the graph tell?
- Whose shoot is the longest?
- Whose bulb grew the most in one week?
- How tall is Andrew’s? Daniel’s? Anne’s?
- How much longer is Tina’s than Mike’s?
- How many are 18 centimeters long?
- Whose bulb grew the most evenly?

Carole also gives the students blank books in which to make drawings and record information about their study. A time line, usually kept on a hallway wall, illustrates the project as the study progresses. Sometimes Carole records the information on the time line, and other times the children do. The time line often includes Polaroid photos and the children’s artwork.

The children record regularly in their bulb books.
Relating math to classroom routines gives children the opportunity to see the usefulness of mathematics in real settings. The following is an example of a math experience that emerged from students needing to have books in which to record classroom activities. Bonnie Tank presented this lesson to first graders in San Francisco, California.

Materials
- 12-by-18-inch construction paper, one sheet per student
- 12-by-18-inch newsprint, several sheets per student

The students in this class were accustomed to making recording books by folding 12-by-18-inch newsprint pages and stapling them inside construction paper covers. They wrote only on the right-hand pages. (Newsprint is thin, so writing on both sides makes student work difficult to read. Also, when students erase, the paper becomes even more fragile.)

Before they made new recording books one day, Bonnie asked the students to figure out how many sheets of newsprint they would each need. “For this book,” Bonnie told the children, “you’ll need 12 pages to write on.”

Bonnie had the children work in pairs to solve the problem. She asked them to explain the problem to each other, figure out an answer, record it
on a sheet of paper, and prepare to explain their thinking to the rest of the class.

All but four pairs of children arrived at the correct answer of six sheets. Of the others, two pairs decided the answer was seven, and one pair decided it was eight. The fourth pair worked on a different problem, trying to figure out how many sheets were needed for all 30 children.

All of the students worked diligently and were pleased with their efforts. After the children had shared their solutions, they assembled their books.

This lesson helped students see that there is more than one way to solve a problem. It related mathematics to a real-life purpose. It also gave Bonnie insights into the children’s individual abilities.

These students illustrated how they would fold each of the six sheets they needed.

This pair of children made an intricate drawing to show their solution.

Tiffany and Michael tried to figure out how many sheets of paper were needed for the entire class.

They decided each child would need 12 sheets.
Sorting experiences encourage students to identify properties of objects, notice their similarities and differences, create classifications, and discriminate between objects. This sorting activity uses letters of the alphabet. Bonnie Tank presented it to first graders in San Francisco, California. Brenda Mercado reported doing a version of the lesson with her multiage primary students in Tucson, Arizona.

### Comparing Alphabet Letters

Bonnie began by writing on the board: B M

“Look at these letters,” she directed the class. “What’s the same about them?”

There was a variety of responses: “Both have a straight line.” “They’re both letters.” “They’re both capitals.” “They’re not numbers.” “Both are in the alphabet.” “Both are not creatures.”

Bonnie then posed another question: “What do you notice that’s different about these letters?”

Again, the answers varied: “One’s M and one’s B.” “One has four lines; one has one line and two bumps.” “The B has curves and the M has straight lines and slanted lines.” “M has the shape of a V, and the B doesn’t.” “M has more lines.”

Bonnie then asked each student to write his or her first initial on a 3-by-5-inch card.

### Materials

- 3-by-5-inch index cards, one per student
organized the students into pairs and had them talk about what was the same and what was different about their letters. Then, in a class discussion, the children reported what they had noticed.

Brandon and Anthony focused on the shapes in their letters. “The B has half circles and the A has a triangle,” they reported.

Zheng and Nektaria discovered a similarity when they rotated their cards. “The Z can turn into an N, and the N can turn into a Z,” Zheng explained.

This sparked a discovery from Hoi Shan and Sammy. “When we turn our initials upside down, they’re still the same,” they reported.

Kimmy reported for her and Jonah: “J comes before K in the alphabet,” she said.

Bonnie then explained to the class that each pair of students was to record how their initials were alike and different. She modeled for them how to fold a sheet of paper in half, label one half “Same” and the other “Different,” write both their names on the paper, and record in complete sentences what they noticed.

The Children’s Work

Zheng and Nektaria repeated what they had reported in class. They wrote: Our names are interesting because the Z can turn into a N and the N can turn into a Z. The N comes before Z and the Z’s sound is like this zzzzzz and the N’s sounds like this nnnnn.

Sammy and Hoi Shan also reported what they had discovered during the class discussion. Under “Same,” they wrote: If you turn it around it is the same.

Sylvia and Katherine looked at various types of similarities between their letters. They wrote: 1. we both have curves. 2. in the alphabet we are 8 letters apart. 3. they are both Capitals.

English was not the first language of many of the children in this class, and this was evident in sentences such as: We don’t have same letters. His is a J and Mys is a M. But all the children’s papers revealed that they seemed to understand what same and different meant, and many were able to use geometric properties in their descriptions.

Zheng and Nektaria’s paper included information about the different sounds their letters make.

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<td><strong>Z</strong></td>
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<td>Same</td>
<td>Different</td>
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<td>Our names are interesting because the Z can turn into a N and the N can turn into a Z.</td>
<td>The N comes before Z and the Z’s sound is like this zzzzzz and the N’s sounds like this nnnnn.</td>
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Sammy and Hoi Shan focused on the lines and curves in their two letters.

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<td><strong>S</strong></td>
<td><strong>H</strong></td>
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<tr>
<td>Same</td>
<td>Different</td>
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<tr>
<td>1. if you turn it around it is the same</td>
<td>1. S has bumps and H doesn’t.</td>
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<td>2. one has curves one doesn’t have curves.</td>
<td>2. H has three lines and S doesn’t.</td>
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<td>3. one have lines and one doesn’t.</td>
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Sylvia and Katherine looked at the letters in various ways.

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<td>Same</td>
<td>Different</td>
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<td>1. we both have curves</td>
<td>1. the k has a line, the S doesn’t.</td>
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<tr>
<td>2. in the alphabet we are 8 letters apart.</td>
<td>2. the S has 4 curves.</td>
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<tr>
<td>3. they are both Capitals.</td>
<td>3. the K has one.</td>
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<td></td>
<td>3. the S and the k are not the number.</td>
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Another Approach

Brenda Mercado does a similar lesson during the first week of school. Rather than have each child write the first letter of his or her name on an index card, she makes cutout letters available to her class. (Brenda has templates for block letters, and parent volunteers cut out a supply.)

“The children can compare their letters more easily when they’re cut out,” she reports. “Also, they can fold them in various ways to find lines of symmetry. And I introduce the idea of closed shapes by having them put a bean on their letters to see if it’s a good fence to keep a dog in the yard.”
Children benefit from activities that ask them to apply what they understand about numbers in situations that are new to them. Students’ responses help teachers assess their ability to use what they know. Bonnie Tank presented this problem to first graders in San Francisco, California.

How Many Dots?

Materials
- One 3-by-12-inch strip of tagboard
- An envelope or covering that the tagboard can slide into

To prepare for the lesson, Bonnie drew 12 dots on a 3-by-12-inch strip of tagboard. She made an envelope that allowed 8 dots to show when she slipped the strip into it.

Bonnie began the lesson by showing the students the entire strip. She didn’t give them time to count the dots. Instead, she slipped the strip into the envelope so that the children could see only some of the dots.

“You can’t see all the dots now,” Bonnie said, “because some of them are hidden. Your problem is to figure out how many dots are on the whole
strip. Because you can’t see the dots that are hidden, I’m going to give you a clue.”

Bonnie took a paper clip. “This dot,” she said, putting the paper clip above the dot she was showing them, “is the sixth dot. Use that information to figure out how many dots there are altogether.”

Bonnie often has students work in pairs or in groups. With this activity, however, she wanted to assess individual students’ understanding. “You’re to work on this problem by yourself,” she said and gave each child an unlined sheet of newsprint. “Put your answer on this paper and also explain how you figured it out. Your explanation is very important because it tells me what you’re thinking. You can also draw pictures to help you figure or explain your reasoning.”

**The Children’s Work**

Children’s responses to the problem differed. Stephanie counted on from the sixth dot and wrote: I think there are 12 dots. I counted the dots from the 6th dot And I counted 12 dots.

Stephanie counted on from the sixth dot to reach the answer of 12 dots.

Carlos solved the problem by trial and error. He wrote: I figered it ote droring and arasing there are 4 insid The envelop there are 12 all twogther.

Carlos used trial and error, drawing, erasing, and counting, to arrive at a solution.

Some students guessed or were unable to explain their reasoning. Most included pictures, some of which were elaborate but had little to do with the problem. A few children solved different problems, figuring out how many dots were inside the envelope or focusing just on the dots they could see.

While the children’s writing revealed the reasoning strategies of individuals, it’s not possible to draw definitive conclusions about a child’s mathematical prowess from just one problem. The information from each problem is valuable when analyzed in conjunction with that child’s responses in other problem-solving situations.
Angie’s drawing didn’t explain how she got her answer.

Eric guessed that there were 11 dots, but he did not explain his reasoning.
Children benefit from problem-solving experiences that help develop their number sense. Christie Brinkley developed this problem for her first graders in Urbana, Illinois. Bonnie Tank presented the problem to first graders in San Francisco, California. She organized the students into pairs and had them explain their solutions in writing.

“Today I have a problem for you to work on with your partner,” Bonnie said. Before presenting the problem, Bonnie described the procedure for how the students would work.

“First, you need to tell the problem to each other to be sure you both understand it,” she explained. “Then, before writing anything, talk about what you’ll write and how you’ll share the work.”

Bonnie then stated the problem: Four raccoons went down to the lake for a drink. Two got their front feet wet. One got its back feet wet. How many dry feet were there?

Bonnie talked about what the students should be writing. “What you put on the paper should help explain your thinking. It can be a drawing, numbers, or anything you think will help. You may also use blocks or counters if you’d like. When you arrive at an answer on which you both agree, put that on your paper as well.”

Before beginning a class discussion about the children’s work, Bonnie gave the students a
direction to help them prepare their presentations. She asked them to review what they had written on their papers.

“Then practice explaining to each other what you did,” Bonnie told them, “so you’ll be ready to share with the whole class. Also, decide how you’ll share. Maybe one of you will hold the paper while the other does the talking, or maybe you’ll decide to share both of those jobs.” Bonnie finds that preparation like this helps students’ presentations go more smoothly.

Jessica and Lisa accidentally drew five raccoons, then crossed out one and wrote Xs on the feet that got wet.

Mohammad and Mark carefully drew two raccoons with their front feet in the water, one with its back feet in the water, and a third outside the water.

Following are similar problems that Christie and Bonnie have given to their students:

• A manufacturer needs wheels for 5 bicycles and 4 tricycles. How many wheels does he need?
• Some children went out to play in the snow. When they went back inside, they put their boots by the door to dry. There were 12 boots. How many children had gone out?
• There were 4 cows and 3 chickens in a field. How many tails and legs were there altogether? (See sample of student work below.)

These two students drew four cows and three chickens, then counted 7 tails, 22 legs, and 29 in all.
The geoboard is an excellent tool for helping students explore shapes and examine their properties. Bonnie Tank taught these two lessons to a class of second graders in San Francisco, California.

Materials
- Geoboards, one per student
- Rubber bands
- Geoboard dot paper, at least one sheet per student (See the blackline master on page 167.)
- One long sheet of butcher paper
- 2-by-6-inch strips of paper, one per student
- Scissors

For a first experience with geoboards, Bonnie gave each student one geoboard and a supply of rubber bands. After giving the children time to explore making shapes, Bonnie asked them to create something that could fly.

“When you find a shape that pleases you,” Bonnie said, “use crayons to draw it on dot paper and then cut it out.” (Transferring geoboard shapes to dot paper is useful for helping young children develop spatial skills and hand-eye coordination.) Bonnie modeled how to do this and then passed out sheets of geoboard dot paper. (See the blackline master on page 167.)
While the students drew their shapes, Bonnie stretched a long piece of butcher paper across the chalkboard to use for a class graph. She also cut a supply of 2-by-6-inch strips of paper.

When the children had finished their drawings, Bonnie began a class discussion. “Edward, what shape did you make that can fly?” she asked.

“A rocket,” Edward replied, holding up his paper.

Bonnie wrote *rocket* on a 2-by-6-inch strip, taped it near the bottom edge of the butcher paper, and told Edward to post his dot paper above the label.

“Did anyone else make a rocket?” Bonnie asked. Several students raised their hands. Bonnie had them go to the front of the room and post their designs, making a column above Edward’s rocket. Bonnie then had the students discuss how the rockets were alike and how they were different.

Bonnie continued this process for all the other shapes the children had made. She made labels for each category and had the children post their shapes, creating a large bar graph. There were 11 categories—rocket, kite, airplane, flying TV, bird, bubble, spaceship, Pegasus, helicopter, butterfly, and bumblebee.

When she finished making the graph, Bonnie said, “Describe what you see.”

“I see that only four people made rockets and six people made butterflies,” said Ana.

“Who can describe what’s on the graph in a different way?” asked Bonnie.

Jesse observed that there were nine kites in the Kites column. Then another child spoke up. “There are seven more kites than spaceships. The spaceships have only two, and the kites have nine.”

During the class discussion, the students talked about differences and similarities among the shapes; which shape they had the most of, least of, and the same number of; how many more rockets there were than birds; and so on. This classifying activity was preparation for a class exploration to focus children on the properties of geometric shapes.

Another Geoboard Lesson

Bonnie began a second geoboard lesson by saying, “I’m going to make a shape on my geoboard that follows three rules: 1. It’s made from only one rubber band. 2. It has no loops, twists, or crossovers. 3. It’s like a fence—it could keep a horse from running away.”

Bonnie made a triangle and reviewed with the children why it followed the three rules.

“Each of you is to make a shape that also follows the three rules,” she said. “Your shape doesn’t have to be a triangle.” She wrote on the board:

1. Only one rubber band
2. No loops, twists, or crossovers
3. Like a fence
When all of the students had made their shapes, Bonnie called for their attention and invited Rebecca to come to the front of the room and show the class the triangle she had made. Bonnie then led a class discussion, asking: “How many sides does Rebecca’s shape have? Has anyone else made a shape that also has three sides? Come up and show it. How are these alike? How are they different?”

The children discussed the similarities and differences among the three-, four-, five-, and six-sided shapes they had made. Bonnie also talked about the names of the various shapes with the students, introducing the proper terminology to describe what they had created.

Bonnie then invited three children to come to the front of the room and show their shapes to the rest of the class.

Bonnie said, “I’d like you to think about what’s the same about all three shapes. First, talk to someone next to you about this.”

After giving the students a minute or two to talk, Bonnie asked for volunteers.

“They all have four corners,” offered Scott.

“They all have four sides,” Jennifer said.

“They all have space in the middle of them,” added Jodi.

“What’s different about all three shapes?” Bonnie asked.

Aaron had an opinion. “Jamie’s isn’t a real shape like those two—the rectangle and the diamond,” he said.

Bonnie continued the discussion, inviting other sets of students to show their shapes to the class.

Note: These lessons appear on the geoboard videotape in the “Mathematics with Manipulatives” series. (See the Bibliography on page 179.)
The story “Jack and the Beanstalk” can be a springboard to a class investigation that involves children in estimating, measuring area, and collecting and representing data. Sharon Dentler taught this lesson to her first graders in Orlando, Florida.

After reading “Jack and the Beanstalk” aloud, Sharon introduced an estimation lesson. She reminded her students that Jack had received only a few beans for the cow. She asked the class, “How many beans do you think Jack would have had if he’d received a whole handful instead of just a few?”

After the children gave their estimates, Sharon asked, “How many large lima beans do you think you can hold in a handful?” Again, the children offered their ideas.

Sharon then gave directions for what the students were to do. “Think about how many beans you think you can hold in a handful,” she said. “Trace one of your hands onto paper, cut it out, and write your estimate on the paper thumb.”

Materials
- The story “Jack and the Beanstalk” (any version)
- Dried beans
- Scissors
- One sheet of chart paper
Sharon modeled this procedure for the class and then continued with the directions.

“Then take a handful of beans and count,” she said. “Work with a partner and check each other’s answer. Then write your actual count on a small paper bean, and glue it to your cutout hand.” Again, Sharon modeled for the children what they were to do.

**A Class Graph**

Sharon gathered the students on the rug in front of the board so they could help her arrange their cutout hands into a graph. She had ruled a sheet of chart paper into vertical columns wide enough for the children’s cutout hands. She wanted to involve the children in making the decisions necessary to create a graph and in considering different ways to group the numerical data.

First, the children identified the smallest number of beans anyone had held—16—and placed that paper hand in the first column. They continued placing their paper hands on the chart, assigning a different number to each column. Counts ranged from 16 to 65, and there weren’t enough columns for all the numbers. (There was some suspicion about the 65, but they didn’t deal with that until later.)

The students then tried grouping the hands by 10s: 1–10, 11–20, 21–30, and so on up to 61–70. Sharon talked with them about eliminating the column for 1–10 since the smallest count was 16. This arrangement fit on the paper, but since more than half of the children’s counts were in the 20s and 30s, their hands seemed too bunched up.

Then a child suggested giving pairs of numbers their own columns—16–17, 18–19, and so on. However, as with the first plan, there weren’t enough columns on the paper. Finally, the class settled on five numbers to a column and labeled the columns 16–20, 21–25, 26–30, 31–35, and so on to 61–65. This approach worked, and the children glued their cutout hands on the chart.

When they looked at the completed graph, the students noticed that some large hands held fewer beans than smaller hands. Sharon asked the students how this could happen, and they came up with a list of possible reasons:

1. Some students could have miscounted.
2. Partners didn’t count because they were in a hurry.
3. The handful was taken with the palm turned down in the bag.
4. The handful was taken with the hand scooping under the beans.
5. Beans that had fallen off the handful on the way to the table had been added to the pile and counted.

**Revisiting the Lesson**

Sharon returned to the activity a week or so later. Children often benefit from trying an activity a second time. Their prior experience eliminates the confusion that new activities often generate and allows children to focus more easily on the mathematics.

Sharon told the students that they would investigate more carefully how the sizes of their hands compared with the number of beans they could hold. They traced their hands again, this time keeping their fingers together. They cut out the paper hands and covered them with beans, using the beans as a nonstandard measure of area. Their areas ranged from 21 to 35 beans. Sharon thought the variation was due, in part, to some of the children’s tracing and cutting skills.

The students again figured out the number of beans they could actually hold. First they set some rules for taking the beans, including making sure partners helped. (The partner of the boy who had recorded 65 the first time confessed that he had never checked.) They recorded the information on their paper hands the way they did the first time, but now their handfuls ranged from 18 to 42.
To help them look at the data in a different way, Sharon listed a pair of numbers on the board for each child—one telling the number of beans that covered the paper hand and the other telling the number of beans in the child’s handful. This information showed that larger hands typically held more beans than smaller hands.

The children wrote about the activity in their math journals. Most wrote about learning that larger hands can hold more beans than smaller hands.

Sharon reported: “Like other estimation activities we had done, the children determined that an estimate was the best we could do to figure out the number of magic beans Jack could hold because we didn’t know the size of his hand or the size of the magic beans.”

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The children wrote about the experience in their math journals. These two students wrote about their discovery that larger hands hold more than smaller hands.

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We did math about that.
We found out that big hands can hold more than loff hands.
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We worked with beans and put them on our hands. Big hands hold more.
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In this lesson, students collect information in three different ways about how many feet there are for the adults living in their houses. The activity gives children the chance to think about numbers in a context related to their own lives. Carolyn Felux taught this lesson to second graders in Converse, Texas.

**Materials**
- Two sheets of chart paper
- Interlocking (Multilink, Snap, or Unifix) cubes, at least 100

“How many adult feet would there be in our room if all the adults living in your houses came to our class at the same time?” Carolyn asked the children.

The children were eager to respond to the question. Their initial estimates ranged from 20 to 10,000, with more than half of the children estimating between 20 and 45.

Carolyn posted two sheets of chart paper and brought out a supply of Unifix cubes. She told the children that each of them was to figure out how many adult feet there were in his or her house. Carolyn also gave the students directions about the three ways they were to report their information. She pointed to one of the sheets of graph paper she had posted. “On this chart,” she said, “record the number of adult feet in your house. On the other chart, write an X to represent
each adult foot in your house. Finally, make a train of Unifix cubes, with each cube representing one adult foot."

Carolyn had several children repeat the directions, and then had the class go to work. After all of the children had recorded the information, Carolyn called them together. She gathered all the trains of Unifix cubes and snapped them together to make one long train on the chalkboard tray. The students were surprised at its length. “Look how long!” “It’s taller than me.” “It almost covers half the board.”

Some of the children wanted to change their original estimates of how many adult feet there would be altogether, some increasing them and others decreasing them.

Carolyn then asked the class, “Do you think we’ll get the same number if we add the numbers on the chart, count the cubes, and count the Xs?” Most of the children were unsure.

“Let’s find out,” Carolyn suggested. She began with the cubes, breaking the train into 10s and then asking the students to count the cubes. There were 70 in all.

Carolyn then moved to the first class chart and used a calculator to add the numbers the children had written. She had a student check off each number as she added it. The total was 86.

Finally, Carolyn drew circles around groups of 10 Xs on the second class chart. There were eight 10s and eight extras—88 in all.

Most of the children were not concerned that the results differed. The few who were troubled couldn’t explain why. Rather than try to resolve the discrepancy at this time, Carolyn ended the lesson by asking the children to continue thinking about the information they had gathered.

**Revisiting the Lesson**

A few days later, Carolyn returned to the data the children had collected. Although the cubes had been returned to their container, the children recalled that there had been 70 cubes in the train. She asked several children to describe what the information on the charts represented.

Carolyn again asked whether the children thought the totals should be the same or different. There were still differences of opinions. A few more students thought the totals should be the same, but many were still uncertain, and others were convinced the results could be different.

“I’m bothered by the differences in our numbers,” Carolyn mused. “They all tell the same thing—the number of adult feet in your houses. I don’t understand how they could be different. What ideas do you have about that?”

“Someone could have put in the wrong number of cubes,” Monica said.

“The Xs looked like more than the numbers,” Tommy said.

“Maybe we counted wrong,” Stenna said.

“Maybe some people counted people instead of feet,” Erin said.

Carolyn then reviewed the chart of numbers by having the children tell their numbers again. This revealed that in one case a 2 should have been a 4, changing the total from 86 to 88—matching the total on the chart of Xs.

April’s observation resolved the discrepancy. “Matthew isn’t here,” she said. The class chart of numbers showed that no one present had recorded eight feet, so that must have been Matthew’s number.

The experience gave Carolyn the chance to talk with the students about several things—how to make sense out of numbers, the importance of checking work, and the benefit of sharing ideas with others.
Sharing an apple equally among three people presents students with a problem that can be solved in different ways and provides a concrete experience with fractional parts. Joan Akers created the following problem for first graders when she was a math resource teacher in Santee, California. Bonnie Tank taught this lesson to first graders in San Francisco, California; Cheryl Rectanus taught it to second and third graders in Piedmont, California.

**Materials**
- Apples, enough for one-third of the students

“\begin{quote} I’ve brought some apples for you,” Bonnie began the lesson, showing the first graders a bag of apples. She removed one apple from the bag and posed a problem similar to the one the students would be solving.

“If I wanted two people to share this apple, what could I do?” she asked.

“You have to cut it,” Matthew said.

“Does it matter how I cut it?” Bonnie asked.

“So we each get the same,” Maria said.

“How much do you each get?” Bonnie asked.

“Half,” several children called out. Though young children haven’t learned the mathematical symbolism for one-half, they’re familiar with the terminology and have had experiences with the concept in different situations.

Bonnie peered inside the bag. “Hmmm,” she said, “I don’t have enough apples for each of you to share with a partner. But I’m sure I have enough apples to give one to every three children.”
Bonnie then introduced the problem the children were going to solve. “I’m going to put you into groups of three. Then you’ll talk in your group about how you might share the apple equally, so you each get the same amount. After you share your ideas, agree on one plan that you think would work. Then write down that plan.”

Bonnie showed the children the knife she had brought for cutting the apples. “When you show me your plan, I’ll follow it and cut your apple, and then you can eat it. It may help to include drawings on your plan so it’s clear to me what to do, but I also want you to explain your plan in words. Also, I’d like you to write how much each person gets.”

“Do we have to eat the apple?” Jason asked.
“No,” Bonnie said, “but I’d still like you to solve the problem of how three people can share one apple.”

Bonnie then reviewed the instructions and wrote on the board:

How three people can share one apple.
Talk
Plan
Write

Bonnie wrote a prompt on the board to help the students as they began writing.

Each person gets _____.

Then she gave each group of three students an apple and a sheet of paper.

Several first graders found it easier to start by cutting the apple into fourths.

The children’s writing showed a variety of methods. Bonnie was not surprised that none of the first graders used the language of fractions to describe how much each child would get. Although experience with the language of halves is common, young children haven’t had similar experience with other fractions.

Albert, Meagan, and Jason each drew a different way to cut the apple into thirds.

The Lesson with Second and Third Graders

When Cheryl Rectanus’s second and third graders worked on the problem, their solutions were more sophisticated. They divided the apples more precisely than the first graders had. Some students considered how to deal with the core of the apple when making equal pieces. Some included thirds or quarters in their solutions.
One group used scratch paper to draw the apple, then wrote extensively about the five methods they had tried for dividing it. They illustrated their fifth method and wrote: *Are last idea looks like a pie, but one piece is cut in 3 pieces.*

These third graders presented five different ideas for cutting the apple. Their scratch paper showed their work.

1. Are first idea was like a pie, but we found out that some pieces were too big. It looked like this.
2. Are second idea looked like stripes, but we found out that someone would get the core. It looked like this.
3. Are third idea was like the letter Y, and that’s are second best one. It look like this.
4. Are fourth idea looks like the letter V, but we found out that some one would get the core. It looks like this.
5. Are last idea looks like a pie, but one piece is cut in 3 pieces. It looks like this.

This group’s plan made sense, but the students’ explanation revealed their confusion with naming fractional parts.

At all three grade levels, the activity provided a way for students to talk and think about fractions and the names of fractional parts in a problem-solving context.
It's important that children experience problems with more than one possible answer, so they can learn that some math problems can have several solutions. David Ott taught this lesson to second graders in Albany, California.

After a storytelling session about safari adventures, David posed a problem to the second graders. “How many animals can there be if there are eight legs altogether?” he asked.

The question gave students a problem to solve that had more than one correct answer. The children worked on the problem in groups of four, describing alternatives and illustrating them.

Heather, Eunice, Aaron, and Rhonda wrote: 

There are two four legged animals, or four two legged animals, or There could be one four legged animal, and two two legged animals.

Matthew, Stephen, Sara Rose, and Kirsten wrote: There could be 8 and 4, 2, 3, 5, 6, 7. We thought this because there could be all kinds of animal. They illustrated seven possibilities.

Stella, Min-Ki, Sherry, and Darren offered two possibilities: Two animals that have 4 legs and 4 animals that have 2 legs.

Dina, Minh, Michael, and Bonnie avoided the mathematical aspect of the problem. They simply wrote: Monkeys, Lions, Girffs.
This group illustrated their examples.

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There are two four legged animals.
or two two legged animals.
or there could have been four legged animals.
and two two legged animals.
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These students figured out seven possible answers to the problem.

This group lost sight of what the animals might be and drew solutions for combinations of one, two, and four legs.
Chapter 5 of *A Collection of Math Lessons From Grades 1 Through 3* (see the Bibliography on page 179) presents four lessons designed to help third graders build their number sense and their understanding of division. David Ott adapted a problem from one of these lessons for second graders in Albany, California.

To present the problem to the second graders, David told them a story, incorporating himself and two boys in the class into the story.

“Craig, Roger, and I were walking to school, and we found 50 cents,” he began.

Several students called out: “Where did you find it?” “Do you come to school together every day?” “Wow, 50 cents!”

“It’s just a story,” David responded.

“You mean it’s make-believe?” Lauren asked.

David nodded, and the class settled down.

He began again. “Craig, Roger, and I were walking to school, and we found 50 cents. We turned the money in to the office. But a week later, no one had claimed it, so we got to keep it.”

“Too bad it’s make-believe,” Roger commented.

David then presented the problem to the class. He asked the students to figure out how much each of the three people would get if they shared the 50 cents equally. “Then write about how you solved the problem,” David said.
The Children’s Work

The children used different strategies to arrive at their solutions and presented different versions of correct answers. Their work is testimony to the uniqueness of children’s thinking and the varied and partial understandings that exist in a class.

Lina and Monica H. wrote: We gave 16 cents to every body and then we saw the 2 more cents were left so we gave them to the teacher. We figured 15 before we thought 16 but then we saw that it was not inof.

Lina and Monica H. verified their answer with an addition calculation.

Monica S. drew three columns and wrote the numbers from 1 to 48, one by one.

Erica and Elizabeth presented a different solution. They decided that each would get 15 cents, and they would give away 5 cents. They provided several alternatives: donating the nickel to the office or the PTA, turning it over to the teacher, or throwing it in the river.

Mia drew marks in three columns and wrote: I put a dot in each column (each column dot’s stand for one cent) until I had put down fifty dots but then one person would only have sixteen cent’s and the other people would have seventeen cent’s. Mia’s writing reflected her enthusiasm for the apostrophe and her partial understanding of its purpose.

Odysseus and William’s method was similar to Mia’s. They drew three stick figures and put tally marks under each. They arrived at an incorrect conclusion, however, but were satisfied with their results. They wrote: We drew three seventeens and it added up to fifty.

Lauren drew 17 squares for each person and checked her answer by adding 17 three times. After figuring that $7 + 7 + 7$ was 21, she made the common error of writing the larger digit in the 1s place and carrying the smaller. This resulted in the incorrect sum of 42. Confused, she tried adding 15 three times, and arrived at 45. That’s as far as she went. Lauren’s erroneous arithmetic is a reminder that we can’t count on the symbolism of mathematics to clarify children’s thinking.
Lauren drew pictures and tried to represent the problem symbolically but became confused.
This lesson gives students the challenge of exploring different ways to divide squares into halves and involves them with geometry and measurement. David Ott taught the lesson to second graders in Albany, California. David followed this lesson with Dividing Cakes (see page 55), which asks students to divide rectangles into different numbers of equal shares. A similar lesson, Cutting the Cake (see page 97), has students divide rectangles into fourths and compare the resulting shapes.

David drew a circle on the board and asked the children how he might divide it in half. After listening to several responses, David drew a diameter of the circle to divide the circle into two equal parts. He shaded one part to model for the children how to show one-half.

David then introduced the activity. “You’re going to experiment with ways to divide squares in half,” he said. He held up a sheet with six squares on it.

“You’ll work in pairs,” David said. “Find different ways to divide each square in half, and then shade half of each square. Be sure that both of you can explain how you divided the squares and why you’re sure each part equals one-half.”

After about 15 minutes, David called the class to attention. “Each pair will show the class

**Materials**

- Paper ruled into six 2 3/4-by-2 3/4-inch squares, one sheet per student (See the blackline master on page 168.)
one square that you divided,” he said. “Pick a square that you think is unique, that nobody else would have.”

As each pair came to the front of the room, David drew a square on the board on which the two students showed their method. The children had to explain why the part they shaded was one-half of the square. David invited others in the class to ask questions and challenge the results if they disagreed.

It was hard for some of the children to duplicate their squares on the board. Sarah and Amika had a particularly difficult time. Once they drew it, however, they were able to explain why what they shaded represented half.

“See, this corner part is the same as this part,” Sarah explained as Amika pointed. “Then these two long pieces are the same, and we cut the middle one in half.”

Jeremy’s example raised the question of how to prove whether what was shaded really was half. He wasn’t sure how to explain it, but he was convinced that he was correct. Children made suggestions, and soon Jeremy was cutting up the square to compare areas.

Monica and Allison explained their drawing by measuring and showing that the line was the same distance from the upper left and lower right corners. Their method opened up many other possible ways to divide a square in half.

David followed this lesson with an exploration of the ways to divide rectangles into 2 through 10 equal shares. (See Dividing Cakes on the facing page.)
In this lesson, students explore how to divide rectangles into different numbers of equal shares. David Ott reported the following experience with second graders in Albany, California. He introduced this activity after his students had completed Exploring Halves (see page 53). In another lesson, Cutting the Cake (see page 97), students divide rectangles into four equal shares, then compare the resulting shapes.

Materials

- 8½-by-11-inch paper, about four sheets per student, cut as described below
- Rulers, one per student

David told the students that they would work in pairs to divide rectangular cakes so that groups of 2 to 10 people would have equal portions. He cut 8½-by-11-inch paper into fourths to make rectangular “cakes” and gave each pair of children about 15 of them.
“Draw lines on each cake to divide it into equal shares for different numbers of people,” he explained. “You may want to fold the paper first. Also, be prepared to explain for each of your cakes why you think the shares are equal.”

As David circulated and observed the children, he overheard some of their comments. “I wish he had given us squares.” “Sharing with five people is real hard.” “Odd numbers of people all seem hard.”

Some of the children folded and then drew lines. Others began by drawing. Some pairs worked together on each cake. Other pairs divided the rectangles and worked individually. David reminded these pairs to discuss their solutions with each other.

One pair of students used a ruler successfully but in an unconventional way. They discovered that the length of a rectangle was exactly the same as five widths of a ruler and used that information to draw fifths!

One pair divided their “cake” as shown below and felt satisfied that there were 16 equal pieces.

“Some are short and fat and some are long and skinny and it all evens out,” the two children explained to David.

“That’s just how it looked to me,” David reported later, “but I had to think a bit to decide how to prove or disprove it.”
Having children write in math class helps them reflect on their experiences and also serves to give the teacher insights into the children’s thinking. In this lesson, Carolyn Felux taught second graders in Converse, Texas, how to play *Roll for $1.00* and then had them write about the game and explain the strategies they used.

**Materials**
- Dimes
- Pennies
- *Roll for $1.00* Directions (See the blackline master on page 169.)
- Dice, one die per group of students

Carolyn organized the class so the students could play *Roll for $1.00* in small groups. “You each need to make a playing board that looks like this,” Carolyn began. She drew an example on the board.

### Roll for $1.00

<table>
<thead>
<tr>
<th>Dimes</th>
<th>Pennies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tr>
</tbody>
</table>
“Use a full page,” Carolyn advised, “and leave enough space so you can fit a small pile of pennies or dimes in each box. Be sure you have seven boxes in each column.”

Carolyn gave the children time to make their playing boards. She chose not to prepare and duplicate playing boards in advance. It’s important that children learn how to organize their work on paper, and this game provided another opportunity for them to practice doing so.

“You’ll play Roll for $1.00 in your groups,” Carolyn said. “The object of the game is to get as close as possible to $1.00 without going over.”

Carolyn wrote the rules of the game on the board and read them aloud.

1. Each person takes a turn rolling one die.
2. On each turn, all players use the number rolled.
3. Each player takes as many pennies OR dimes as the number rolled. A player may not take both pennies and dimes on the same turn.
4. Each player puts pennies in the Pennies column and dimes in the Dimes column.
5. Whenever a player has 10 or more pennies, he or she MUST exchange 10 pennies for a dime. Put the dime in any box in the Dimes column.
6. Players who go over $1.00 are out of the game and must wait for the next round.
7. The game is over after seven rolls. The winner is the player who has the closest to but not more than $1.00.

Carolyn distributed a die and a supply of pennies and dimes to each group. She also gave each group a copy of the directions. (See the blackline master on page 169.) As with all new activities, there was confusion during the first round of the game. Carolyn gave assistance as needed—answering questions, explaining again, and resolving differences. The children soon were able to play easily.

Writing About the Game

After the students were familiar with the game, Carolyn asked them to write about their experiences. She was interested in having them evaluate the game and also explain the strategies they used when playing.

“There are two parts to this writing assignment,” Carolyn told the children. She wrote prompts on the board to help the children get organized:

1. I think the game is ____________________.
2. The best way to play the game is ________________.

Most of the children reported that they liked the game. Liz’s response was typical. She wrote: I think the game is fun because you get to play with money and you get to count it.

From April: I think the game is a good game because if you know how to count money it will help you practice.

Stenna wrote: I think the game is fun because it is a learning game.

Sarah had a complaint. She wrote: I think the game is noisy because the children play with the money on the desk!

Clint panned the game. He wrote: I think the game is boring because all you do is roll and put on money. In response to the second question, however, he had a clear and concise strategy: The best way to play this game is you count every time. When I rolled a big number I put pennies and when I rolled a low number I put dimes.

Greg gave the game a mixed review. He wrote: I think the game is good and bad because I like to count but then you get stuck you have nine pennies and 90 but I can’t go over ten pennies and also I can’t go over a dollar. Greg’s idea about how to play the game was operational rather than strategic. He wrote: The best way to play the game is to count the money every few minutes so that you don’t get over ten pennies or ten dimes so if you do you can trade in the pennies for dimes.
Grades 2–4: Roll for $1.00

Greg had mixed feelings about *Roll for $1.00*.

![Roll for $1.00](image)

I think the game is fun because it’s a learning game.

The best way to play the game is to use dimes for 1s, 2s, 3s, and 4s, and pennies for 5s and 6s.

Liz combined both approaches in a lengthy response: The best way to play the game is to count the money every few minutes so that you don’t get over ten pennies or ten dimes so if you do you can trade in the pennies for dimes.

Stenna’s strategy was to use dimes for 1s, 2s, 3s, and 4s, and pennies for 5s and 6s.

April’s idea was similar. She wrote: The best way to play the game is to use dimes when you roll a little number like 4, 3, 2, and 1. And when you roll a big number like 5 and 6 you use pennies if you want to.

Erin took a broader focus in her response. She wrote: The best way to play the game is to first learn the rules and don’t go over a dollar. You use dimes and pennies. I like the game because it is a thinking and a money game too because I like playing with money a lot. How I win on big rolls I put pennies and on little rolls I put dimes. I like to buy stuff with money like gum toys and presents the best. Money is important to me because you need it to buy clothing, shelter and food.
Sometimes the inspiration for a math activity comes naturally from conversations with students about things in which they’re interested or involved. In this case, a third grader in Mill Valley, California, brought to school a rubber band ball she had made and was showing it to other students. I used the opportunity to focus the children on estimation, ratio, proportion, and patterns.

Materials
- Rubber bands, at least 600

Brandie had brought a rubber band ball to school, and I overheard her tell the other students at her table that her father had given her the rubber bands. She had worked on the ball for two days and used 320 rubber bands.

“I have an idea,” I said to Brandie and the others at her table. “How about showing the ball to the rest of the class without saying how many rubber bands you used? Then they can guess.”

I called the other students to attention, and Brandie showed them the ball. “When you make a guess,” I said, “Brandie will tell you whether your guess is too large or too small. Listen to all the guesses and to Brandie’s clues. They can give helpful information for figuring out how many rubber bands there are in all.”

Most of the children were eager to guess. Few of them, however, made use of the clues. For example, Michael guessed 400, and Brandie told...
him that was “too high.” Then Vanessa guessed 500, and everyone looked expectantly at Brandie for the clue. It was only after about 10 guesses that some children started to notice that some guesses weren’t useful or necessary.

As the guesses got closer to the correct number, I began recording them on the board. (I think I should have recorded the guesses and clues from the start, as seeing the numbers seemed to be useful for some of the children.) Finally, Timmy guessed correctly.

I then measured the diameter of the ball for the children by propping a chalkboard eraser on each side of the ball and measuring the distance between. It was 3½ inches. The word diameter was new for the children, so I added it to our class list of geometry words.

“I wonder what the diameter would measure,” I said to the children, “if we added more rubber bands to the ball.”

Brandie’s hand shot up. “My dad said he has 300 more rubber bands that I could have,” she said excitedly. Brandie is a quiet child who generally doesn’t ask for or receive a great deal of attention. She was enjoying center stage right now and was eager for it to continue.

“Suppose Brandie added 300 more rubber bands, 100 at a time,” I said to the class. “After adding 100 more, how many rubber bands would there be on the ball?” Patrick answered that there would be 420.

“And then 100 more?” I asked. Grace answered with 520.

“And the last 100?” I asked. I received a chorus of 620. I recorded on the chalkboard:

<table>
<thead>
<tr>
<th>Rubber Bands</th>
<th>Diameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>320</td>
<td>3½ inches</td>
</tr>
<tr>
<td>420</td>
<td></td>
</tr>
<tr>
<td>520</td>
<td></td>
</tr>
<tr>
<td>620</td>
<td></td>
</tr>
</tbody>
</table>

“With 100 more rubber bands,” I asked the class, “what would the diameter measure?”

Timmy guessed 5 inches, but couldn’t explain why. Erika guessed 6 inches, and then Michael guessed 7 inches; neither had an explanation.

Grace thought it would be 4½ inches. “I figured that it took about 100 rubber bands to make an inch,” she explained, “so another inch would make it 4½.” Grace’s idea made sense to others who nodded in agreement.

Patrick had a different thought. “I don’t think it would get so big,” he said, “because when you put rubber bands on, they have to stretch more because the ball is bigger, and when they stretch more they’ll be thinner. So I think it’ll be 4 inches.” This idea appealed to some of the other children.

I asked Brandie to bring in 100 rubber bands, so others could help her add them to the rubber band ball.

### The Next Day

Brandie came to school the next day with a plastic bag of 1,000 rubber bands. “My dad said I could have these because it was for math,” she announced.

I gave the children at Brandie’s table the task of counting out three piles of 100 rubber bands and adding the first 100 to the ball. When they were finished, I called the class to attention and measured the diameter. It measured 3¾ inches.

“Does anyone have an idea about why the diameter is less than you predicted?” I asked.

“I think it’s like what Patrick said,” Jill answered. “They stretch more because the ball is bigger, so it didn’t grow much.”

Bayard, who had helped put the extra rubber bands on, had another idea. “I think these rubber bands are thinner than the others,” he said.

“With this new measurement,” I then asked, “think about what the diameter will measure with 100 more rubber bands. Tomorrow we’ll discuss your ideas, add more rubber bands, and measure again.”

We took the time the next few days to continue the investigation. I purposely spread the activity over time to allow the children to reconsider their ideas. Too often, we race for answers and push for conclusions. Missing is time for musing, reflecting, and letting ideas sit. Over several days, more of the children used reasoning techniques and made predictions they could explain. The time was well spent.
These two problems require students to apply number and measurement skills as well as their understanding about the calendar. I got the idea for the lesson from Marge Tsukamoto, a teacher in San Francisco, California. I presented it to third graders in Mill Valley, California, at the beginning of the school year.

"How many days have you come to school so far this year?" I asked the third graders. The students weren’t sure. We looked at the calendar together and verified that this was the eleventh day of school.

I took a 1-by-12-inch strip of paper ruled into 1-inch squares and taped it horizontally at the top of the chalkboard. With a marker, I numbered the first 11 squares.

"Why do you think I numbered up to 11?" I asked.

Laura volunteered that it was because they’d been in school 11 days so far.

"I have two problems for you to solve in your groups of four," I then told the class. "First, I want

Materials

— One 1-inch strip of paper, ruled into 1-inch squares

“H”ow many days have you come to school so far this year?" I asked the third graders.

The students weren’t sure. We looked at the calendar together and verified that this was the eleventh day of school.

I took a 1-by-12-inch strip of paper ruled into 1-inch squares and taped it horizontally at the top of the chalkboard. With a marker, I numbered the first 11 squares.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

“Why do you think I numbered up to 11?" I asked.

Laura volunteered that it was because they’d been in school 11 days so far.

“I have two problems for you to solve in your groups of four,” I then told the class. “First, I want
you to figure out how many days you’ll come to school this year. Next, I want you to decide where the strip will end if we continue numbering it for all the days of school until summer vacation.”

The Children’s Work

The discussions in the groups were extremely animated. Children brought different bits of information and approaches to their groups. There was disagreement in some groups about how many days were in a year. Some knew exactly, some had a close idea, and some had no idea at all.

Most of the groups figured that they came to school for nine months and focused on how many days were in each month that they came to school. One girl started to write down the “30 days has September” poem to help her group figure the differences among months. One group figured that there were either 20 or 21 school days in each month. But Teddy knew something about February and said they should count only 18 days for it.

There was much adding and figuring on scratch paper. Then, when I reminded the students about writing down their ideas, they scurried to record their thoughts.

Only three groups tackled the problem of how far the strip would stretch. They used rulers and yardsticks and relied on visual estimating.

One group had three students in it: Patrick, Mairead, and Bryce. Patrick was working separately from the other two. He concluded that there were 213 days of school, while the other two thought there were 192 days. However, Patrick marked a spot for the strip to end that wasn’t as far around the room as the spot that the other two had marked.

“That can’t be,” Mairead said, “because your guess for the number of days is higher.”

These three students wrote the following about their different answers: Patrick thinks there are 213 days in school. Bryce and Mairead think there are 192 days in a school year. Patrick thinks it is 213 because it is not a hole year. Bryce and Mairead think it is 192 because it sounds like the rite number.

The numbers in the problem posed difficulties for the children. Still, they were interested and engaged in thinking and reasoning. This was a good reminder that even when a complicated problem poses difficulties for children, if their curiosity is engaged, the problem can provide a valuable experience.

Both of these groups mentioned removing vacations and weekends, but each came up with a different answer.

Bryce and Mairead estimated 192 days. Patrick disagreed, believing there were 213 days in a school year.
The Place Value Game has long been one of the standard activities I give to students of all ages. It blends probability with place value in a game that has an element of chance and gives children practice reading large numbers and writing decision-making strategies. I taught this lesson to third graders and sixth graders in Mill Valley, California.

**The Lesson with Third Graders**

The object of the Place Value Game is to try to make the largest number possible with the digits determined by a die or spinner. I introduced a four-digit game to third graders, but it’s possible for younger children to play with two- or three-digit numbers and for older students to play a five-digit game.

Although I planned to have the children play the game in groups of four, I introduced the game to the entire class. I drew on the board a game board and asked that each student copy it on a sheet of paper.

**Materials**

- Dice or spinners with numbers 1 to 6, one per group of students
“Make the boxes large enough so that you can write a numeral in each one,” I said. “And draw the game board near the top of your page so that you can fit many games on one sheet of paper.”

I purposely did not prepare a worksheet for the students to use. I think that children benefit from having the responsibility to organize their own recording, and this game gave them practice doing so.

“To play the game,” I said, “I’ll roll the die, and you’ll write each number that comes up in one of the boxes in your game board. Once you write a numeral, you can’t move it to another box. The object is to end up with the largest number possible.”

“Why do we have a reject box?” Andreas wanted to know.

“I may roll a number that you don’t think will help you get the largest number possible,” I answered. “So you have one chance to reject a number. But the rule is the same: Once you write a numeral in the reject box, you can’t change it.”

I know to expect confusion the first time I try any new activity with students. “Let’s just try a game,” I said, “and we’ll see if that helps you understand how to play.”

I rolled the die five times, stopping each time to give all of the students a chance to record the number. There were groans and cheers when I rolled a 6. I already had rolled a 5, and some students had written it in the 1,000s place, while others had held out, hoping for a 6.

At the end of the game, I asked students to read their numbers aloud to the others at their tables. This would give them practice reading numbers in the thousands. “Each of you should read your number aloud,” I said. “Then the person with the largest number should raise his or her hand.”

“We had a tie,” Jill said.

“Then both of you can raise your hands,” I responded. I called on several children to report their results. I recorded their numbers on the board and reviewed the names of the 1s, 10s, 100s, and 1,000s places with them.

After playing one more game with the class, I had the students play in small groups. The children liked the game, and after they had played it on and off for several days, I initiated a class discussion to hear their ideas about how to win the game.

The children offered a variety of approaches: “It’s a good idea to put a 6 in the first box.” “I always put a 1 in the reject box.” “I put a 5 or 6 in the 1,000s place.” “You have to decide if you’ll take a chance with a 3.”

I didn’t comment or probe their ideas but merely gave all who volunteered the chance to report.

“When people play games,” I said, “they often have a strategy, a plan for deciding what to do.” I wrote the word *strategy* on the board. I then explained that I wanted the students to write about the strategies they used to decide where to write numbers as they came up.

“Think about how you’d give someone else advice about how to win,” I said.

Grace developed a strategy for playing the game but was also philosophical about winning and losing.

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**Grace:**

Our table got 9 all ties and 8 all in a row. One of them was all 1s. My strategy was that if there was one I put it in the reject box. I put 6s in the thousands box, I put 3s in the ones box, I put 4s in the tenths box, I put 5s in the hundreds box. I never won alone but I was in a few winning ties and a few losing ties. I was disappointed when I lost but I know games are for fun not winning.
Teddy thought it was a good idea to take chances in the game.

I thought taking chances on low numbers like 2 is better than putting it in the reject box. That’s how I won a game. We all almost had the same strategy. That’s how we got all ties.

I lost three games from not taking chances. So, I changed my strategy.

The Lesson with Sixth Graders

After I explained the rules to the sixth graders, I rolled the die, listing the numbers on the board after each roll. After all of the students had filled in their boxes, it was clear to them from the numbers on my list what the largest number could be. Some students were pleased that they’d made choices that turned out to be advantageous; others were disappointed in their choices.

I played the game several more times with the class, and then I described their assignment. “Work in your groups, and write a decision-making strategy that you think would give the best chance of producing the largest number possible,” I said. “Imagine that you’re programming a computer to play the game. Describe how you would tell the computer to decide where to place each number as it comes up.”

After the students wrote and shared their strategies, they were interested in doing some comparison testing with them. We played several games as a class. I rolled the die, and each group used its strategy to decide where to record the number. Doing this pointed out to some students that their strategies weren’t complete or clear enough, and some groups had to revise their directions. Also, the students decided that it would be necessary to play many games before they had convincing evidence that some strategies were more effective than others.

As an extension, some students wrote strategies for two-, three-, and five-digit games. There was a good deal of discussion among students about which game was easiest to win.

These students wrote a step-by-step procedure for playing the game.

1. Rolls of 1 and 2, put in reject box; if reject box filled, put farthest right as possible.
2. In rolling a six, put it farthest left as possible.
3. 5 in first roll or three on first roll go in 100s place.
4. 4 on second roll goes in 10s place.
5. On 4th roll, 5 goes to thousands.
6. 4, rolled after a four in 10s place, put in 1000s place.
7. Rolling a 1, 2, or 3, other than 1st roll, put farthest right as possible.

After explaining how to play the game, this group described its strategy.

Roll the die -

You must put the number you roll in a box and you may not change it to a different box after you roll again. Put the 6 in the top left box. If you get a 5 or a 4 put them in the hundreds or tens box. If you get a 3, 9, or 1 put them in the ones box or the reject box. If the space for a 1, 2, or 3 is taken move it to the nearest right hand box. If the space for a 4, 5, or 6 is taken move it to the nearest left hand box.
Making classroom graphs gives students experience with collecting and interpreting statistical data. This lesson uses children’s interest in their names and their classmates’ names to help them learn to create a graph, make generalizations from the data, and think about number relationships. A similar activity gives older students the opportunity to think about fractional relationships. I taught the lesson to second graders in Mill Valley, California and to fifth graders in San Francisco, California.

I began the lesson with second graders by ruling four columns on the board and numbering them 1, 2, 3, and 4. I wrote Marilyn in column 3.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Marilyn</td>
<td></td>
</tr>
</tbody>
</table>

“Why do you think I wrote my name in column number 3?” I asked.

Several students had theories. “You wrote it there because you have three names—a first...
name, a middle name, and a last name,” Marisa said.

“It’s true that I have three names,” I replied, “but that isn’t why I put my name in column 3.”

Sara had a different idea. “Maybe it’s because you are 30 years old,” she said.

I informed her that I was not 30 years old, and my age had nothing to do with my reason.

Edward spoke next. “Because it’s your favorite number,” he suggested.

I told him that wasn’t my reason either. Marie came up with the correct answer. “Is it because your name has three syllables?”

I was surprised by Marie’s response. I didn’t expect the children to figure it out—or know the word syllable.

“Yes, that’s why,” I responded, without revealing my surprise.

I then said my name, clapping each syllable as I did so. “My name has three claps. That’s three syllables.”

I wrote above the columns:

How many syllables are in your name?

I erased my name and had each student in turn come up, say his or her name, clap for each syllable, and then write the name in the correct column.

Encouraging Mental Calculation

I stopped after about half of the class had recorded. To engage the students in calculating mentally, I asked them to look at what had been written so far.

“How many people on the chart have one-syllable names?” I asked. There were four. We then counted and found that there were nine two-syllable names and three three-syllable names. I wrote 4, 9, and 3 on the chalkboard.

“Let’s figure out how many names are posted so far,” I said. “How much is 4 plus 9?”

About half the children immediately raised their hands. “Say the sum together,” I said, and received a chorus of “13.”

“And 13 plus 3?” I continued. The class answered “16” in unison.

“Since there are 27 of you altogether, how many more names need to be written on the chart?” I then asked. There was some hesitation. After a few moments, three children raised their hands, though tentatively. After a few more moments, one more child raised her hand. I had the four volunteers report, and I received different answers: 9, 12, 11, and 13.

Next, I asked the students who had not yet written their names on the chart to raise their hands. We counted and found that 11 children still had to record.

We completed the chart. There were 7 names in the one-syllable column, 13 in the two-syllable column, 7 in the three-syllable column, and zero names in the four-syllable column.

Analyzing the Data

I wanted the children to draw conclusions from the information posted, so I gave several examples.

“Let me tell you some things I notice from examining this chart,” I said. “There are more names in the two-syllable column than in any of the other columns. Also, zero children have four-syllable first names.” The students nodded.

“I’m interested in what you notice,” I then said. “Who can tell us something else from looking at the information on the chart?”

Edward spoke first. “All the children have different handwriting,” he said.
Though the children had indicated their agreement with my generalizations, they were much more enthusiastic about Edward’s idea. “Look Allison’s name is big.” “Marisa used cursive.” “Phillip’s name is the smallest.”

Sara had a different thought. “Timmy could’ve written his name in the 1, 2, or 3 column, but I could only write mine in the 2 column.”

Again, the children were interested and talked about how Timmy could have written his name as Tim or Timothy. They found others with that same option—Nick, Doug, Michael, Edward, Phillip. Doug wanted to move his name and write it as Douglas instead.

Suzanne added, “All the names that can be written in more than one column are boys’ names.”

Nick focused on another aspect of the names. “Marisa was the only one who wrote her name in cursive,” he said.

This was not going in the direction I had hoped for. I was looking for generalizations from the numerical relationships on the graph. My plan was to use these generalizations to create word problems for the students to solve. However, their generalizations were obviously more interesting to them than mine, so I let them continue.

After all of the children had had a chance to share their thoughts, I wrote two problems on the board:

1. Do more or less than half the children have two-syllable names? Explain why.
2. How many more children have two-syllable than one-syllable names? Explain why.

I asked the students to write group responses. They were accustomed to doing this in math class. One child from each group got a sheet of paper, and the children negotiated who would do the writing. I was interested in their responses, as we had done no formal work with fractions.

All but one group answered both questions. That group spent a lot of time copying the chart. They answered the first question but ran out of time and steam.

These groups had no problem answering the two questions about the name graph.

1. The 2-syllable column has thirteen, and the 1 column and three columns have 7 each. 7+7=14 and 13 is not half the class, 14 is greater than 13. It is less.
2. Column 2 has 6 more than column 1. 7+6=13. If you add one more it will be 14.

The answer is less. We think it is less because 7+7=14 and there is 13 in the two syllable group.

2nd one
6 people. We think this because 13-7=6.

There is 7 in the 1-syllable column and 7 in the 3rd column. Less than half because 7+7=14 and there is 13 2-syllable names.

6 because there is 7 in the 1-syllable column and 13 in the 2-syllable column and 6+7=13.
The Lesson with Fifth Graders

At the beginning of class one day, I wrote on the board:

How many syllables are there in your first name?

Under the question, I drew four columns, numbering them 1, 2, 3, and 4. I modeled what I meant by syllables by saying my name, Marilyn, and clapping for each syllable. Then I wrote my name in column 3.

“I’d like each of you to come up and write your name in the correct column,” I said. The children in this class sat in groups of five or six, and I called the students to the board in groups.

There was much discussion and questioning about what they were to write. “Should I write Matt or Matthew?” “Most people call me Jerry, but my real name is Gerald, and some people call me Jer.” “I can’t decide whether to write Jennifer or Jenny.”

I repeated several times that the choice was theirs, but that they each could sign only once.

After all of the students had recorded, I focused them on interpreting the information on the graph. We counted how many people had signed in each category, and I recorded these counts at the bottom of each column. There were 9 names with one syllable, 14 with two syllables, 6 with three, and 1 with four.

“I’m going to ask you some questions that you’re to answer using the information on this chart,” I told the students. “When I ask a question, please don’t call out the answer or raise your hand to reply. Rather, I’d like you to put your heads together in your groups and quietly discuss what you think the answer is. When you’ve done this, raise your hands to show you’re ready with a response.”

I then asked my first question. “How many people signed their names on the chart altogether?”

After a moment, all hands were raised. “When I count to 3,” I said, “I’d like you all to say the answer together, softly.”

I counted, and got a chorus of 30.

“How’s my next question,” I continued. “Do more or less than half the names have two syllables?” Some of the children immediately raised their hands, so I reminded them, “Talk about this in your groups, raise your hands when you agree on an answer, and then I’ll call on someone to respond.”

After a few moments, I called on Irene. “Less than half,” she said.

“Can you explain why you think that’s true?” I asked.

“Because 14 and 14 add to 28, and that’s less than 30,” Irene explained.

Sean raised his hand. “I figured it another way. Half of 30 is 15, and that’s more than 14,” he said.

“Any other thoughts?” I asked.

There were none, so I asked another question, writing it on the board as well:

Do more or less than ⅔ of the names have more than one syllable?

This seemed easy for the students, and their explanations were similar to the ones they had given for the first question.

I then asked, “Do more or less than ¼ of the names have three syllables?” I wrote the question under the first question on the board.

This also seemed easy, and students again offered two explanations.

David said, “It’s less because 6 times 4 is 24 and that’s less than 30.”

Geneva said, “It’s less because if you divide 30 by 4, you get . . . ” After hesitating, she added, “more than 6.”

“How much is 30 divided by 4?” I asked.

Matt answered, “Seven and some left over.”

I continued with several more questions of this type, each time having students mentally figure answers and explain their thinking. The discussion gave students practice with calculating mentally, contributed to their learning about fractions, and helped their development of number sense.
The National Council of Teachers of Mathematics’ position statement, “Calculators and the Education of Youth,” recommends that calculators be used “at all grade levels in class work, homework, and evaluation.” Because my third graders in Mill Valley, California, had limited experience with calculators, I planned some introductory explorations.

Materials:
- Calculators, at least one per group of students
- Calculator Explorations (See the blackline master on page 170.)
- NCTM Position Statement (See the blackline master on page 171.)

“How many of you have calculators at home?” I asked the class. Most of the students raised their hands.

I then held up one of the calculators we had in the classroom.

“What number do you think you’ll see on the display if I press 2 plus 2?” I asked. The children answered “4” in unison.

I pressed 2 + 2 and asked the children sitting directly in front of me to report what number they saw on the display.

“It says 2,” Chris said, surprised.

“It must be broken,” Andreas said from the back of the room.

“No,” I answered, “I just haven’t asked it for the answer yet.”
Mairead’s hand shot up. “You have to press the equals,” she said.

I did so. Now Chris reported that he saw a 4. Everyone seemed relieved.

“What I’m going to do now,” I said, “is press equals again and again. Each time I press the equals key, Chris, Jason, and Brandie will report what they see.” I did this and the three students called out the numbers as they appeared—6, 8, 10, 12. As they continued, all the children began chiming in—14, 16, 18. I stopped and called the class to attention.

“How can those of you who are sitting in the back of the room know what numbers are coming up when you can’t see the display on the calculator?” I asked.

“Easy,” Erika said. “They go by 2s.”

“Ahh,” I said, “you’ve noticed a pattern. That’s just what you’re to do when you explore the calculators—look for patterns.”

I showed them how the “C” button cleared the display. I then asked another question, “If I press 3 plus 2, what will be on the display?”

Some said 5; some answered 2; some said to press equals.

I pressed the buttons 3 + 2 =, and 5 was displayed.

“Now watch what happens,” I said, “when I press the equals button again.”

“You’ll get 10,” Ann called out.

I pressed it.

“No, it’s a 7,” Chris said.

I pressed the equals sign several more times, and the children seated in the front of the room reported—9, 11, 13, 15.

“What’s the calculator doing?” I asked.

“It’s adding 2 again,” Teddy said.

I cleared the calculator.

“This time,” I told the class, “I’ll start with ‘2 plus 3 equals’ instead of ‘3 plus 2 equals.’ What pattern do you think will be displayed as I keep pressing equals, equals, equals?”

The children had various ideas. Rather than answer them, I suggested that they work in groups to find out.

I then showed them the sheet of explorations I had prepared. (See the blackline master on page 170.) I explained to the students that when they saw three periods ( . . . ), it meant “and on and on.”

---

The Children’s Discoveries

The children made a variety of discoveries. One group wrote: We learned that if you press 2 + 2 it equals 4 but if you keep pressing the equals sign it will keep adding by twos. The next thing we learned is that when you add something like two plus three the last number that you add is the number that will keep adding together when you press the equals sign. I think the reason it always adds the last number when you press the equal sign more than once is because it is the number you add on to the first number.

Another group, however, was confused by what happened. They wrote: We found out if we press 2 + 3 10 times it will add up to 32. And 3 + 2 will = 23. We thought it would come out the same because it was the same numbers. We pressed 2 + 2 10 times. It came out 22. We pressed it again it came out 22. We think it is weird because we pressed 3 + 2 and it came out 23 and we pressed 2 + 3 and it is the same numbers. Others, we pressed 8 + 9 10 times. It came out 98. We pressed 9 + 8 it came out 89. We thought it would come out the same numbers.

---

This group noticed that the calculator repeats the second number in a two-addend addition problem.

```
Plus Patterns
2+2=4=6=8=10 It goes by twos
3+2=5=7=9=11 It goes by twos
2+3=5=8=11=14 It goes by threes
The second number in the problem is the one you add on.
6+3=9=12=15=18 It goes by threes
3+9=12=21=30=39 It goes by nines
8+6=14=20=26=32 it goes by sixs
```
A third group related addition to multiplication. They wrote: We found out about patterns when we pushed $2 + 2$ patterns then we pushed $=$ and it went $2, 4, 6, 8, 10$, and so on and then it's just like multiplication because when we pushed the $3's$ it went like $3, 6, 9, 12$ and so on and the $4's$, $5's$, $6's$, $7's$, $8's$, $9's$, $10's$, $11's$ and $12's$ and they all went in a pattern.

Included in another report was an explanation of the associative principle: When you add three numbers it adds the first two numbers together. Then it adds the answer and the last number. Then when you keep pushing $=$ it will add the last number.

One group found: Odd + odd is always even. Even + even is always even.

These students discovered that if they continued to press the minus button, the calculator would go beyond $1$ and show negative numbers.

These students found an addition pattern that they “tried with hundreds and it worked.”

### Plus Patterns

Odd + odd is always even. Even + even is always even. We did $8 + 8$ and $8 + 12$. It always adds up by the last number. The second number is always the one you count by, like six plus $4$ counts by four. The same thing with $1, 2, 3, 5, 6, 7, 8$ and $9$. It works with any number. Jill and Laura tried with hundreds and it worked.

### Minus Patterns

We found out that if you press an odd number and an even number and press the equals sign, it will go down to one and then if you press the equals sign some more, it will start going down to negative numbers. The negative numbers are when after you get down to one and then keep pressing the equals sign, it will look like it went back up to $26$ but it is really going down to negative numbers because if you look at the side of the number there is a tacaway and that means that those numbers are numbers below zero.

For another group, however, numbers just got smaller and then larger: When you do $20 - 2$ after $0$ it starts counting up to as high as you want to. And when you do $20 - 2$ you go down by two and go up by two.

These introductory activities provided just what I had hoped for. The children were involved.
While learning about a useful tool, they also were exploring mathematical patterns. In addition, from watching the children, listening to their discussions, and reading what they wrote, I learned more about each student’s thinking process and understanding.

Some children brought in calculators from home. Some of these calculators, however, didn’t have the capability of adding a constant when the equals button was pressed repeatedly. This difference fascinated the children.

A Note About Communicating with Parents

It’s important that we communicate with parents about what their children are experiencing and learning in classroom mathematics instruction. I duplicated the NCTM position statement (see page 171) and sent it home. Also, I asked the children to share with their families the patterns they discovered.
One recommendation in the NCTM Curriculum and Evaluation Standards for School Mathematics calls for having students “apply mathematical thinking and modeling to solve problems that arise in other disciplines” (p. 84). Joanne Curran made this happen with her third graders in Olivette, Missouri. The following describes five days of activities in her class during which she integrated the children’s math work with their study of pioneers.

**Materials**
- Post-it Notes, 1-by-2 inches, at least two per student
- Paraffin wax, one 10-pound block
- Two 5-pound coffee cans
- Candlewick string
- One large pot
- One hot plate (or stove burner)
- Pot holders
- One box, approximately 1 cubic foot

“How many dips do you think it would take to make a candle with a base that measures 1 inch in diameter?” Joanne asked the class. The children had been studying pioneers and had learned about life in log cabins.

Joanne gave the children Post-its on which they wrote their estimates. The children went one at a time to the board and posted their estimates, which ranged from 100 to 300 dips, with estimates close to 250 being the most common.
Joanne had assembled the materials needed for each child to make a candle. She filled the large pan about half full of water and put it on the hot plate to heat. Using a hammer, she broke the wax into chunks and put them into one of the coffee cans. Then she put the can in the pot of water to melt the wax. She filled the other coffee can with cold water.

Joanne showed the children how to make their candles.

“First you cut a 6-inch length of candlewick string and tie one end of it around a pencil,” she explained.

“Then you dip the string in the wax and slowly count to 10.” She demonstrated.

“Lift it out,” she continued, “and immerse it in the cold water to speed up the cooling process. Dip again and again, until the candle is the right size.” To help the children get a sense of the process, Joanne repeated the procedure several more times, dipping her string into the hot wax and then plunging it into the cold water.

She organized the children into pairs. She explained that partners would take turns, one dipping while the other kept track of the dips by tallying.

“When the diameter of the base measures 1 inch,” Joanne said, “record the number of dips on a Post-it. Then switch jobs and make another candle. When you’re finished, the next pair goes.” She gave each pair a number, so all the students would know the order of their turns.

“Save your Post-its,” Joanne concluded, “so when the entire class is finished, we can post the data and compare them with our estimates.”

Joanne went on with her day’s plan of activities, and pairs of students made their candles throughout the morning.

Later, when all the candles were made and the students had posted their data, Joanne led a discussion about the range of the data of their actual dips, what occurred most often, and how the two sets of data compared. She also talked informally about the average number of dips, referring to what was “most typical.” (The number of actual dips ranged from 30 to 60, with close to 50 being most common.)

Day 2

Joanne read to the students an article in the *St. Louis Post Dispatch* reporting that a candle could light approximately 1 cubic foot. She had brought in a box about that size to show the children, then posed a problem for the class to investigate.

“How many candles do you think a pioneer family would need to light a log cabin?” Joanne asked the class.

The children worked on this problem in groups of four. Most groups drew pictures of a one-room log cabin, decided on the lengths of the walls, and figured the area of the floor. A few groups tried to figure the volume of the cabin as well. One group drew a rectangle and divided it into small squares, each representing what a cubic foot box would occupy on the floor of the cabin.

“I wasn’t looking for any single right answer,” Joanne reported. “I was interested in the students’ thinking processes and methods for solving this problem.” Joanne had the children share their ideas in a class discussion.

Lindsey pointed out that pioneer families didn’t really need to light the entire log cabin.

“There’s always a fire in the fireplace or the stove,” she said, “and that gives light.”

Scott had a different idea. “They would only light part of the cabin at a time,” he said, “and carry the candles with them when they had to go to a darker part of the cabin.”

Doug pointed out that they needed more candles in the winter when it was darker longer.

“They wouldn’t need so many in spring and summer,” he added. This gave Joanne the opportunity to talk with the children about why days are different lengths in summer than in winter.

“How many candles would they need to use each night?” Derek wanted to know.

Adam suggested that they burn one of their hand-dipped candles to see how long it lasted. Joanne agreed. The class went on to other work while the candle burned. It lasted for 2 hours and 45 minutes.
Day 3

After reviewing the previous day’s experience, Joanne posed another problem. “How many candles would a pioneer family need for a full year’s supply?” she asked.

Joanne listed three assumptions on the board.

1. The family’s one-room log cabin was no bigger than 20-by-20 feet.
2. The family had candles about the same size as the ones the students had made.
3. The family would light only the part of the cabin they were using.

The class measured 20 feet by 20 feet in the classroom to get a sense of the size of a log cabin. Also, the children thought they should say that the candles lasted about three hours, since candles might be different.

Groups approached the problem different ways. One group said it got dark in winter at about 5 p.m., and the family would need to burn two to three candles each night until 8 p.m., when they’d all go to bed. They figured that Ma needed two candles near her to light 2 cubic feet of space so she could sew, while Pa needed only one candle to read because he sat near the fire.

Then the students were stuck. Finally, they decided to multiply 365 by 2½. Not knowing how to multiply by fractions, they multiplied 365 by 2 and then added ½ of 365 to get 914 candles. (Later they lowered their estimate, reasoning that in the summer the family needed less than one candle per day since it stayed light until 8 p.m.)

Another group used a similar line of thinking, but decided that in the winter the family also needed half a candle each morning. They wrote: *They got up when it was still dark and needed light to dress and eat breakfast.*

A particularly energetic group divided the year into four parts with 91 days in each. (They ignored the extra day.) They decided the family needed one candle per day in summer, two per day in spring and fall, and three per day in winter. They arrived at a total of 728 candles for the year.

Two groups divided the year into winter and summer, figured the number of hours per day they needed candlelight in each half year, and used that information to estimate a year’s supply of candles.

Three groups came up with answers between 900 and 1,000; two groups estimated 528 and 789. Joanne instructed the groups to record their thinking processes. She found their methods remarkably complex. One group recorded 27 steps they had used; another recorded 12.

“The children were pleased and congratulated each other profusely,” Joanne reported.

Day 4

Joanne began the lesson by telling the students that it had taken the class more than three hours to make 20 candles. “If the pioneer family needed between 500 and 1,000 candles each year, how many hours would they need to make them?” she asked.

The children worked in the same groups as before. Some groups found the problem easy; others were confused. One group figured the time for 500, 750, and 1,000 candles, not sure how many the family would need. Two groups divided 1,000 by 20 to get 50 but then forgot to multiply this by the three hours it took them to make the 20 candles.

Two other groups figured the total hours and divided by 24 to find the number of days. Another group contested this approach “because they had to sleep and nobody works for 24 hours straight.” Then the students chose to divide by 10. “It’s easier and rounder,” they said, “and pioneers had to work at least 10 hours a day.” In situations such as this, calculators allow children to work with numbers that otherwise might be out of their reach.

Day 5

Joanne gave the class the problem of finding a quicker and easier way to dip the candles. “Pioneer women had dozens of jobs to do each day,” she told the children, “and couldn’t spend
all their time dipping candles! How could they approach this problem?”

Surprisingly to Joanne, this problem was the most difficult for the students to solve. Only one group came up with an option that they felt good about. They wrote: *Get a long stick and a huge pot of wax. Tie about 20 strings on at one time and get two people to dip for a day.*

**A Final Note**

Joanne felt that the time for these activities over the five days was well spent. The children were involved, interested, and engaged in thinking mathematically. They were pleased with their work—and so was Joanne.

Joanne presented one additional problem to the children: If they had made 12-inch candles instead of 6-inch candles, how would their answers change? This problem encouraged the children to reexamine their work and reflect on their thinking.
**The Game of Leftovers**

*Math By All Means: Division, Grades 3–4*, which I co-authored with Susan Ohanian (see the Bibliography on page 179), is a five-week replacement unit. Lynne Zolli taught the unit to third graders in San Francisco, California. Here is an abbreviated description of a whole class lesson that uses Color Tiles to give students experience with division and remainders.

_I’m going to teach you how to play a game called Leftovers,_” Lynne announced to her third graders. “It’s a game of chance for partners and uses remainders. The winner is the person who gets more leftovers. I’m going to play the game with Irene, so you can see the way it works.” Lynne invited Irene to join her at the board.

Lynne cautioned the class, “The game isn’t hard to play, but you have to count carefully and keep careful records.” She picked up a plastic...
cup containing tiles. “Your first job is to make sure you have 15 tiles. Irene, will you make sure we have 15?” Irene counted the tiles and nodded her head.

“Also, you need a die and six squares of paper,” Lynne said, showing these items to the children. “We’ll call these squares ‘plates.’” Lynne turned to Irene and said, “You go first. Roll the die.”

Irene rolled a 4. Lynne directed Irene to lay out four plates and divide the tiles among them. “Be sure you put the same number of tiles on each plate,” Lynne said.

Irene first put 2 tiles on each plate. After counting what was left in the cup, she put 1 more tile on each plate. “There aren’t enough to go around again,” Irene said. “There are 3 left over.” Lynne illustrated on the board what Irene had done.

“Next, both people record the division sentence,” Lynne said and wrote the equation $15 ÷ 4 = 3 R3$. Then she wrote the letter I in front of it.

“This I will help me remember that Irene rolled the die,” Lynne explained. Lynne designated a place on the board for Irene to record, and Irene copied what Lynne had written.

Irene rolled the die, so she gets to keep the 3 leftovers, but she puts the rest back in the cup,” Lynne said. “How many tiles do we have now?”

“Fifteen!” exclaimed Reggie. “We have 15 altogether,” Lynne said, “but how many are in the cup now that Irene got to keep the 3 left over?”

“Now it’s my turn to roll the die,” Lynne said. She rolled a 6, put out six paper squares and divided the 12 tiles among them. She then illustrated on the board 2 tiles in each of six squares.

“There’s nothing left over,” Brittany said.

“That’s right,” Lynne said. “Who can tell us what to write?” She called on Amari. She and Irene recorded as Amari dictated: $12 ÷ 6 = 2 R0$. Then she added a Z in front to indicate that it had been her roll.

Next, Irene rolled a 5. She put out five plates and divided the tiles as Lynne drew on the board. Aaron dictated the equation, and Lynne and Irene both recorded it: $12 ÷ 5 = 2 R2$.

Lynne and Irene continued to play the game, recording the plays on the board. When all the tiles had been distributed, the game was over.

Lynne and Irene continued to play the game, recording the plays on the board. When all the tiles had been distributed, the game was over.

“Who won?” Truc wanted to know.

“Let’s each count the tiles we kept.” Lynne said. Irene reported that she had 9, and Lynne reported 6.

“Do we have all 15?” asked Lynne. Irene counted the tiles; others figured in their heads or used their fingers.

“You and your partner have to do one more thing before you start another game,” Lynne said. She posted a large sheet of chart paper and titled it “Division with R0.”

“On this chart, write all the sentences from your papers that have a remainder of zero,” Lynne instructed. “But don’t record any that are already on the chart.” Lynne recorded the four different sentences that had remainders of zero.
The students were accustomed to working with partners, and they went right to work.

### Observing the Children

Lynne circulated among the pairs. When children rushed through the game, she tried to slow them down, encouraging them to be methodical in laying out the paper squares and counting the tiles they had at the start of each round.

Matthew and Irene were typical of the children who valued speed over care. Both confident of their mathematical ability, they didn’t share their tiles on the paper squares and didn’t count the tiles at the beginning of each round. They relied on thinking they knew the right answers to the division problems. Lynne scanned their paper.

“You’ve made two errors,” she told them.

“Where?” Matthew demanded to know.

“You can find them if you replay the game,” Lynne said. She sat with them as they got started, checking each of their sentences with the tiles.

When Matthew found the error he had made dividing 14 into four groups, he said, “Oh, no. Then it all changes from here.” Lynne agreed. After reminding them to slow down and be careful with their calculations, Lynne left them to redo the rest of their game.

### A Class Discussion

The next day, Lynne called the class to the meeting area at the front of the room. “When you play Leftovers, you have to have a remainder to get out of a number,” she said. “What was a very hard number to get out of?”

Demetrius and Wesley exclaimed together, “Twelve! It’s impossible.”

Lynne copied onto the board division sentences from the class chart that began with 12:

\[
\begin{align*}
12 \div 6 &= 2 \text{ R0} \\
9 \div 3 &= 3 \text{ R0} \\
6 \div 3 &= 2 \text{ R0} \\
6 \div 6 &= 1 \text{ R0}
\end{align*}
\]

“The only thing left to throw is a 5,” Lynne said and added \(12 \div 5\) to the list on the board.

“What happens with this number?” she asked.

“It’s 2 remainder 2,” Matthew said. Lynne completed the last sentence:

\[
12 \div 5 = 2 \text{ R2}
\]
“Wow!” Demetrius said. “Every number on the die except 5 comes out even, with no remainders. That’s what makes it so hard.”

Lynne later introduced an extension of the game—Leftovers with Any Number. In this game, students play Leftovers with one difference: They choose how many Color Tiles to start with. Giving students the choice is a way to allow them to gauge their own comfort level with numbers.

When playing Leftovers with Any Number, Samantha and Juliette were surprised by how many times they rolled before getting a remainder for 20.

<table>
<thead>
<tr>
<th>Leftovers with 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6 \div 4 = 1 R0$</td>
</tr>
<tr>
<td>$8 \div 5 = 1 R3$</td>
</tr>
<tr>
<td>$10 \div 2 = 5 R0$</td>
</tr>
<tr>
<td>$10 \div 3 = 3 R1$</td>
</tr>
<tr>
<td>$12 \div 2 = 6 R0$</td>
</tr>
</tbody>
</table>
One day, I purchased a gift for a friend and brought it to the gift wrapping desk. While I waited, I watched the two women working there wrap several presents. Their spatial skills were impressive. Each would pick up a box, tear off paper from a huge roll, and have just the right amount. Then they would cut a length of ribbon that, again, was just what they needed.

“How do you know how much to cut?” I asked the woman who took my box.

“I just judge, I guess,” she answered with a shrug.

“Are you usually close?” I asked.

“Pretty much,” she said. “I get a lot of practice.”

As she wrapped my gift, I wondered about the ribbon. I couldn’t even imagine how much ribbon I would need for the bow, much less the whole package.

Months later, Bonnie Tank and I were talking about lesson ideas for third graders. Bonnie wanted problems that would involve children with two-digit numbers.

“How much ribbon does the bow on a gift take?” I asked her.

“How big a bow?” Bonnie asked.

“A nice bow,” I answered.

We both made estimates. We each got ribbon and tied a bow. Our thinking about how much ribbon we needed inspired the following problem-solving lesson, which involves third graders with geometry, measurement, and number. There’s no end to the real-world sources for problem-solving ideas!

Bonnie taught this lesson to third graders in Mill Valley, California.
“**Materials**

- Boxes of various sizes, one per group of four students
- Yarn, one yard-long piece per group
- Ribbon or yarn (Use yarn that doesn’t stretch much.)

**We’re going to investigate how much ribbon you would need to wrap a present,** Bonnie told a class of third graders. She showed the class one of the boxes she had collected.

“Let’s start by thinking about the bow,” Bonnie went on. “In your groups, estimate how much ribbon you think you would need just for the bow. Discuss your estimates in your group.”

Bonnie gave a yard-long piece of yarn to each group. “We’re going to use yarn instead of real ribbon,” she explained. “Use this yarn to tie a bow around a pencil. It’s up to your group to decide what size bow you’d like and how long the ends should be. After you make the bow and trim the ends, measure the yarn you used.”

The children were interested in the problem. There was a great deal of discussion. Some children got rulers. Some made estimates with their hands. After a little more than 10 minutes, all groups were ready to report. Bonnie had one child in each group hold up the bow as another in the group reported. Bonnie recorded the lengths on the board:

- 12 inches
- 19 inches
- 26 inches
- 19 inches
- 24 inches
- 13 inches

“What do you notice about these lengths?” Bonnie asked the class.

There was a variety of responses: “They’re all in the teens or in the 20s.” “None are more than 30 inches.” “None are more than 26.” “All are more than 10.” “Most of them are more than 12.” “They go even-odd-even-odd-even-odd.” “Two are 19 inches.”

Because of an assembly, the math period was cut short. Bonnie continued the lesson the next day.

“Today,” Bonnie began, “each group will get a box to wrap.” Bonnie had a box for each group of four children. The collection included two shoe boxes, one box large enough for a shirt, one box that had held a ream of paper, the box that held the class Color Tiles, and two small cartons.

“Your group’s job is to figure out how many inches of ribbon you need to wrap your box, including the bow.” Bonnie demonstrated how to wrap the yarn around in two directions with a bow on top.

Then Bonnie explained to the children how they were to work. She wrote the directions on the board and elaborated on them:

1. **Talk about a plan to solve the problem.**
2. **Decide on the length of yarn you need.**
3. **Write about how you got your answer.**
4. **Measure and cut yarn to test your answer.**

**The Children’s Work**

There was much talking, measuring, drawing, writing—and thinking. All but one group reported that their estimates were short. For example, one group drew an illustration of their box and explained clearly how they made their estimate. They wrote: *This is how we figured it out we measured across the bottom and doubled that then we measured the side and doubled that then we doubled it again and then we added 19 inches for the bow. We tested it and it barely fit we think we measured it well, but we needed one or two inches more.*
These students made a careful estimate but found that they needed 1 or 2 inches more ribbon.

Another group whose estimate was too short explained: *We mesherd the top, botom, and the 4 sides. Then we added 22 for the bow and we got 69. We tested our ribbon and we where serpriste because it was totaly small. We suspktid it was going to be big!*  

One group dealt with the dilemma of too little ribbon by wrapping the package around the corners instead of the way Bonnie had demonstrated. They wrote: *We tested our ribbon and it was just right for us or We think we should of had 10 more inches for the regular way.*

The activity engaged the children and gave them valuable hands-on experience with estimating and measuring.
In 1994, I taught a unit I had developed on probability to a class of third graders in Mill Valley, California. This was the third time I had taught a version of this unit, and I felt ready to write the replacement unit *Math By All Means: Probability, Grades 3–4*. (See the Bibliography on page 179.) Following is an abbreviated description of *Match or No Match*, one of the menu activities.

**Match or No Match**

I think it’s valuable, as often as possible, to relate a new activity to children’s real-life experiences. Therefore, before introducing this activity to the class, I talked with the children about how they decide among themselves who gets to pick first or go first in a game.

The students reported several different ways they used. The most common was playing ro-sham-bo (their name for Scissors-Paper-Rock). Other methods included tossing a coin and calling heads or tails; one person putting an object in one hand and the other person guessing which hand it’s in; and both people showing one or two fingers, with one person winning if there’s a match and the other if not.

“Although you have different ways of choosing,” I said, “it seems to me that they’re all the same in one way. They each give both people the same chance of winning. They’re fair games...
because both people have an equal chance. In this activity, I’m going to explain three ways to choose, and you’ll investigate whether or not each version is fair.”

As the students watched, I put two blue tiles and one red tile into a paper lunch bag. “Suppose you and your partner decide to choose who goes first in an activity by reaching into the bag and drawing out two tiles,” I said. “One of you is the match player and the other is the no-match player. When you look at the two tiles you remove, the match player wins if they’re the same color, and the no-match player wins if they’re different colors.”

To demonstrate for the class, I asked Lori to join me at the front of the room. “Would you rather be the match player or the no-match player?” I asked her. She chose match.

I asked Lori to reach into the bag and remove two tiles. She did so and held them up for the class to see—a red and a blue.

“I’m going to record what happens so we can begin to collect some data,” I said. I wrote Match and No Match on the board and made a tally mark next to No Match.

I asked Lori to replace the tiles. I shook the bag, explaining to the class that I did this to make sure I had mixed up the tiles. Then Lori drew again. Again, she got one of each color. I made another tally mark. We repeated this a third time with the same result. The students began discussing whether they believed match or no match was better.

“The game that Lori and I were playing is one of three versions that you’ll investigate,” I said. “Each version is played the same way, by drawing two tiles out of the bag and seeing if they match or not. The difference in the three versions is what you put into the bag.” I listed on the board:

<table>
<thead>
<tr>
<th>Version</th>
<th>Tiles</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 Blue 1 Red</td>
</tr>
<tr>
<td>2</td>
<td>2 Blue 2 Red</td>
</tr>
<tr>
<td>3</td>
<td>3 Blue 1 Red</td>
</tr>
</tbody>
</table>

I explained to the children that they were to put the tiles in the bag and take 20 samples. Then they were to write about whether they thought the game was fair, or take more samples and then write.

After the students had explored all three versions of the game, I initiated a class discussion. The discussion was animated, and students argued about the fairness of the different versions. We examined the data they had collected, determined how many matches and no matches were in each version, and then talked about the fairness of each game.

**Writing About the Game**

After several days of discussion, I asked the children to write about the game. I circulated as the children worked. I noticed that some children drew conclusions from the data they collected, without thinking about the mathematical theory. Mercedes, for example, wrote the following: *I think for version 1 it’s fair because we got 10 matches and 10 No matches. We thought it was really weird but that’s just what happens. I think for version 2 it’s fair also because there were 2 blues and 2 reds and match and no match got 10 and I knew that it would be fair because 2 blues and 2 reds sound like they would be fair. I think for version 3 match is going to win because there are one red and 3 blues and on our sheet it’s 9 to 11 and no match wins. My data doesn’t make any sense.*

When I asked Mercedes why her data didn’t make sense, she replied, “That’s not what I thought was going to win.”

I sorted the students’ papers into three piles: those who were able to analyze the versions of the game completely and correctly (11 children), those whose reasoning was partially correct (10 children), and those who didn’t seem to have much understanding at all (4 children). I noticed which children were able to list the possibilities and use the information to decide about the fairness of the game; who relied primarily on the data, either on the class chart or from his or her own individual experiments; and who wasn’t able to think at all about the mathematics of the situation.
Mercedes drew conclusions from the data she collected, without thinking about the mathematical theory.

Abby’s explanation was clear and succinct.

Elliot wrote only about version 1 and did not deal at all with the probability aspect of the game.

A Mathematical Note

Of the three versions of the game, only version 3 is mathematically fair. This seems counter-intuitive to many children (and adults), and the data from your class may or may not support the conclusion. It’s important to remember that collecting a larger sample of data provides more reliable information. Also, although data are useful for thinking about and discussing the game, data are not sufficient to prove a theory.

One way to analyze version 1 is to list the three possible combinations of drawing two tiles:

- B1, B2—match
- B1, R—no match
- B2, R—no match

This shows a 2/3 probability of getting no match and a 1/3 probability of getting a match. (Try analyzing version 2 this way to prove that it’s more likely to get no match than match.)

Version 3, with three blue tiles and one red tile, disturbs some people when they analyze the possible combinations. Listing the possible
pairs of tiles you can draw reveals that half of them produce a match and half produce no match.

B1, B2—match
B1, B3—match
B2, B3—match
B1, R—no match
B2, R—no match
B3, R—no match

Some people aren’t sure that the six pairs are really equally likely and think that the blues make a match much more possible. I believe that the six possibilities above represent all the ways and that the game is fair, but you’ll have to think about it to convince yourself.
Inspiration for problem-solving lessons comes from many sources. I got the original idea for this lesson from Jason, a third grader in Mill Valley, California. (A variation of the second activity in this lesson appears on page 141.)

One day, in response to one of my math-is-everywhere discussions with the class, Jason said, “We could do a math problem about the ceiling. We could figure out how many holes there are in the ceiling.” The ceiling was covered with 12-inch-square acoustical tiles, each with a pattern of holes in a square array.

“How could we figure it out?” I asked.

“We need to know how many tiles there are,” Michelle answered.

All eyes were drawn to the ceiling. The children began to count. It was a funny sight—27 little children looking at the ceiling intently, their heads bobbing as they counted. They counted, disagreed, counted again, argued, and finally came to an agreement. There were 32 tiles counting one way and 24 the other way.

“Now you multiply,” Laura said. And though others nodded in agreement, no one was able to explain why multiplying made sense. However, the children used their calculators to find that there were 768 tiles. Some checked by adding 24 32 times or vice versa.
“Now we need to know how many holes in one tile,” Patrick said.

None of us, however, could face counting the holes in one tile from our vantage points. We’d had enough for one day.

I was able to get an extra tile from the custodian and brought it into class the next day. This made the problem more accessible. The tile had 23 holes on each side. The children used their calculators again, first to find that each tile had 529 holes and then to calculate that there were 406,272 holes in the ceiling. They seemed proud of their work, even though they weren’t able to read the number. I had them write about what they had done.

Michelle, Mike, and Tim wrote about how to figure out the number of holes in the ceiling.

**Holes in the Ceiling**

First count the numbers on one side. Then count the numbers on the other side. Multiply the two numbers. Repeat the same for the ceiling. Multiply the number you got on the ceiling with the number you got from the tiles, and the answer will be 406,272. That’s what we got.

**The Around-the-Edge Problem**

“I have another problem for you to solve,” I said. “I’ll call it the ‘around-the-edge-problem.’ How many holes do you think there are around the edge of each tile? You could count them, but I’d like you to think about other ways to figure it out. Talk about this in your group and see if you can agree on an answer. Then each of you write about your thinking.”

The groups’ discussions were animated. I was surprised and delighted by the variety of approaches that emerged. For example, Grace wrote: I think there are 88 holes around the edge. I think this because if you add 23 + 23 + 23 + 23 you get 92 but that is not correct because you are counting the corners twice. I thought if you could pretend the corners weren’t there and you added what was left which is 21, 4 times, you get 84. Then you add the corners on and you have 88.

Grace suggested pretending that the corners weren’t there, then adding them back in later.

Marina found a different way to approach the problem: I think there are 88 holes around the edge. I think this because if you take away the corners there will be 21 holes on each side and then you add one corner to each side and will have 22 holes on each side. 22 + 22 + 22 + 22 = 88.

Teddy wrote: I think there are 88 holes around the edge because there are 23 holes on one side and you can’t count the hole on the corner twice so it would be 22 on two sides and on the last one, it would be 21 because you can’t count the same one twice on both corners. If you add 23 + 22 + 22 + 21, it equals 88.
Ann explained: I think there are 88 holes around the edge because there are 23 holes on two sides and on the other two sides there could not be 23 because the two end holes would be already counted so I subtrakted 23 tackaway 2 and I got 21 so there is twenty one holes on the other two sides so I added 23 + 23 + 21 + 21 and I got the number 88.

Nick had a variation on Ann’s method. He wrote: There are twenty-three holes across the top and bottom and twenty-one along the sides. I added twenty-three plus twenty-one and I got forty-four. I timesed that by two and got eighty-eight.

Nick counted the corners only for the top and bottom edges.

\[
\begin{array}{c}
23 \\
+ 21 \\
\hline
44 \\
\times 2 \\
\hline
88
\end{array}
\]
This geometric problem gives students experience with exploring the fractional concept of fourths and with measuring to compare areas. Carolyn Felux developed and taught this lesson to fourth graders in Converse, Texas. Two similar lessons are designed for younger students: Exploring Halves (see page 53) has students divide squares into halves, and Dividing Cakes (see page 55) has students divide rectangles into different numbers of equal shares.

**Materials**

- 8½-by-11-inch sheets of paper

Carolyn asked the fourth graders to explore ways to cut a cake into four equal pieces. She gave the children 8½-by-11-inch sheets of paper to use as “cakes,” and she asked them to sketch the different ways they found to divide them.

All of the children quickly found three ways to cut the cake into fourths.

Some children explored further and found additional solutions.
This activity is not unlike textbook problems that ask students to divide shapes into fourths. However, Carolyn extended the task by asking the children to respond in writing to the following direction: “Take one piece from each cake you cut. Compare the pieces to see if each one gives the same amount of cake as the others. Explain your reasoning.”

The children’s papers revealed that almost half of them thought that fourths in different shapes were different sizes. These children did not understand that all pieces that are one-fourth of the same whole are the same amount.

Megan wrote: I do not think the pieces are the same size because they aren’t the same shape.

Susan wrote: No, because some of them are short. And some of them are long. Because they are not all the same size.

Others, however, understood that the pieces were the same size. Brandon explained how he verified his thinking with the pieces. He wrote: Yes I do think they are all the same amount of cake. Why I think that is because I measured them or in other words I investigated. I cut one to measure the other one so that it fit it right.

Sara wrote: yes, I do think they are the same because all of the objects or shapes are ¼. No matter if you stretch & pull they are still the same size. But if you cut the cakes into smaller pieces of course they will not be ¼ any more.

Sara understood that the shape of the ¼ piece was not important.

Carolyn might not have been aware of the students’ misconceptions if she hadn’t asked them to write about their thinking. With this information, she knew to provide additional investigations with geometric shapes to give the students opportunities to see that different shapes can have the same area. Having students make rectangular “cakes” with 24 Color Tiles, for example, can help them see that it’s possible to build different shapes of rectangles with the same area. Cutting out tangram puzzles and learning that the square, parallelogram, and middle-sized triangle are all the same size also connects to the cake-cutting activity. Experiences with fractions in contexts other than geometric shapes are also important. With all activities, assessing what children truly understand is a key element.
Making generalizations—one of the important higher-order thinking skills—is difficult for some children. I derived this lesson from one of my long-time favorite activities— “The Consecutive Sums Problem”—which appears in A Collection of Math Lessons From Grades 3 Through 6. (See the Bibliography on page 179.) I taught this lesson to a fourth grade class in Bellevue, Washington.

The fourth graders were seated in groups of four and were accustomed to working cooperatively. To begin the lesson, I wrote five sets of numbers on the board:

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>42</td>
<td>43</td>
<td>44</td>
<td>45</td>
</tr>
<tr>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
</tr>
<tr>
<td>77</td>
<td>78</td>
<td>79</td>
<td>80</td>
</tr>
</tbody>
</table>

“Each of these rows is a set of four consecutive numbers,” I told the class. “What do you think I mean by consecutive numbers?”

The students described consecutive numbers in several ways: “They go in order.” “They go up by 1s.” “They don’t skip any.”

“When I examine these sets,” I went on, “I notice that when I subtract the first number in each sequence from the last, I always get an answer of 3. For example, in the first one, 4 minus 1 equals 3. In your group, check to see if you agree this is true for all of these sequences.”
While the students were talking this over, I wrote the generalization on the board:

**The difference between the first and last numbers in a sequence of four consecutive numbers is always 3.**

The students agreed that what I said was correct and that what I had written also expressed this characteristic.

“What I would like you to do in your groups,” I then explained, “is to see what else you can say about sets of four consecutive numbers. What you are looking for are characteristics that hold for all sets. Write sentences to describe your group’s generalizations.”

Table 2 wrote three generalizations that used addition and one that used multiplication.

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. If you add the first and the last, you always get an odd number.</td>
</tr>
<tr>
<td>2. If you add the two middle numbers, you always get an odd number.</td>
</tr>
<tr>
<td>3. If you add all four numbers, you get an even number.</td>
</tr>
<tr>
<td>4. If you multiply the first and the last number, you get an even number.</td>
</tr>
</tbody>
</table>

The students went right to work. After about 15 minutes, I asked for their attention and told each group to choose one person to read one of the group’s generalizations.

“I’ll go around the room,” I said. “Each group will report just one conclusion on a turn. Listen carefully to the other generalizations because I want you to read one that hasn’t been reported.”

After each group read a statement, the others were instructed to check to see if it matched one they had written and to talk about whether they agreed it was true. All six groups had different statements to report, and their statements stimulated new thinking.

Table 4 looked for patterns within the display of the four numbers.

<table>
<thead>
<tr>
<th>Table 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>If you always start with an even number, the last number will be odd. And if you always start with an odd number, you come out with an even number. Every number next to each other, if you subtract them, you’ll get the #1.</td>
</tr>
</tbody>
</table>

Throughout the lesson, I used the words conclusion, characteristic, and generalization as often as possible. Student understanding develops from hearing words in context. Also, students benefit from experiencing many such investigations, which support both their number sense and their ability to make generalizations.

**Additional Explorations**

Following are other explorations that engage students with making generalizations.

1. What can you say about any 2-by-2 array of numbers on a 0–99 chart? What about a 3-by-3 array?
2. What can you say about any three diagonally adjacent numbers on a 0–99 chart?
3. Try problems 1 and 2 but use numbers on a calendar instead. Do your generalizations still hold? Why or why not?
I found *The Firehouse Problem* in a lesson in “The Whole Language Connection,” an article by Nancy Casey in the 1991 edition of *Washington Mathematics*, a publication of the Washington State Mathematics Council. Nancy Casey describes the whole language approach as one in which teaching is “organized around meaningful creative work by students.” In the article, she describes four days she spent in a second grade classroom, “squinting over the wall that divides mathematics from language arts” to explore how the whole language approach to teaching reading and writing might transfer to the teaching of mathematics. I taught the lesson to third graders in Mill Valley, California.

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**Materials**

- The Firehouse Problem recording sheet, at least one per student (See the blackline master on page 173.)
- Cubes, tiles, or other objects, at least seven per student (optional)

I held up a copy of the recording sheet and pointed to the pattern on it. “This is an unusual map of a town,” I explained. “On this map, the lines represent streets, and the dots represent street corners. All the houses in this town are located at corners, and there’s at least one house at each corner.”

I then presented the problem. “The town needs firehouses,” I said, “and the mayor has said that if a house catches on fire, fire trucks shouldn’t have to drive more than one street to get to it. It’s important that every house be on the same corner as a firehouse or only one street away.”

To check that the children understood, I chose one dot as a possible firehouse location and
asked which houses would be guarded by that station. After a few more demonstrations with different dots, all the children seemed to understand the parameters of the situation.

"Firehouses are expensive, and the people in the town would like to have some money left over for some swimming pools," I continued. "The problem is to figure out the fewest number of firehouses they can build and where to build them."

The students were fascinated by the problem and eager to get to work. I gave them a few more directions before distributing the recording sheet.

"Use cubes, tiles, or other objects to mark where you think firehouses might go," I said. "When you’re satisfied with a solution, record it by circling the dots where the town should put firehouses." I also told the students that they could work individually, with a partner, or in a small group. They dove into their explorations.

This problem is from the area of graph theory. The map is a graph, but a different sort of graph from the types commonly used to represent a collection of data. This map comprises a set $V$ of vertices (the dots) and a set $E$ of edges (the lines). The solution—the fewest possible firehouses that meet the criteria—is called the minimum dominating set, and the only reliable way to find it is trial and error. However, there are many possible strategies for making guesses.

**Observing the Children**

As the children worked, I circulated, talking with them about their discoveries. Their theories were plentiful. Some looked for locations that serviced the most houses. Some noticed that a number of houses were particularly difficult to service. Some explored the different effect of locating firehouses near the outskirts of town or in the middle of town.

When students presented a solution to me, I gave them the choice of looking for a different solution or writing about the strategies they used to think about the problem. As students began finding solutions with the same number of firehouses, I had them compare their work to see if they had located the stations at the same corners.

At the end of the period, I asked the children to report what they had discovered so far. When we finished our discussion, no one was sure that the class had found the fewest number of firehouses possible, and all of the students were interested in continuing with the problem.

The next day, I started the class by having a few volunteers explain the problem for the two children who had been absent the day before. I also told the entire class that at some point they would have to write about their thinking. During this second class, some children continued to look for different solutions, even if they didn’t find fewer firehouses, while others began writing about their strategies.

**The Children’s Work**

Most of the students wrote about trying one strategy that wasn’t successful, then finding another that they thought was better. For example, Gabe found a way to place seven firehouses in the town and believed his strategy would help him reduce that number to six. He wrote: The way I figerd out how to do seven fire stations first by using Patricks idea that coverd five squars then I put a firehouse on the top of the triangle on the bottom right, and so on. I [In] the beginning I didnt have a stratogy, but twords the end I started to pick up a
stratogy I think you should put more on the side than in the middle. Now I’m pretty shere I can do six I’ve almost got it.

Gabe found a strategy he liked and believed he could use it to place even fewer firehouses.

Bridget and Norah worked together. Bridget wrote: At first Norah and I first would only put down a Fire Station if it would take care of four or five house, but we didn’t get very far that way. Then we realized that there were some places where fire house could only take care of one or two house but we need to put them down our some house wouldn’t get a Fire Station. The lest Fire Stations we could get is seven.

Bridget described the strategies that she and Norah used.
Most of the children continued working on the problem for a third day, and some stayed engaged with it even longer. Some made up their own “maps” to make firehouse problems for others to solve. After three days, I asked the children which grade levels they thought this problem was suitable for. After some discussion, the conclusion was unanimous that it was good for grades 3 through 8.

“It might be hard for second graders,” Erin added, “but I think they could handle it.”

Erin was pleased with the strategy that she and Jill used.

Firehouse Problem

Jill and I got seven firehouses on the paper. Our strategy was to look for a place where there were most houses around one spot. There was one house that did not get covered so we had to put a firehouse there. But our strategy worked.
Students in the intermediate grades are comfortable with the idea of what a square is. This activity gives them the challenge of examining their understanding more deeply and applying it to a problem-solving situation. It also helps students become familiar with the properties of other shapes and how shapes relate to one another. I taught this lesson to fourth graders in Mill Valley, California.

The Largest Square Problem

Materials
Three paper shapes, each made from a sheet of 8½-by-11-inch paper as described below, one of each per student, plus extras

One right isosceles triangle made from folding and cutting a sheet of 8½-by-11-inch paper as shown.

One pentagon
I began the lesson by giving each student a sheet of 8½-by-11-inch paper.

“Fold the paper to make a square,” I said. “Do this so you make the largest square possible.”

All the children got involved with the problem. As I expected, some were able to do this more easily and quickly than others. When I noticed that almost all of the students had finished, I asked for a volunteer to demonstrate a solution to the class. Marina did so, showing the class how she first folded a corner of the paper down to make a triangle and then folded up the flap.

“How can you be sure your shape really is a square?” I asked.

Marina didn’t have an immediate answer, but several other students raised their hands.

“You could use a ruler,” Jamie said.

“What would you do with the ruler?” I asked her.

“I’d measure it,” Jamie responded.

“What would you measure?” I probed. Several other students raised their hands, but I let Jamie continue.

“I’d measure the sides to make sure they’re the same,” she said.

“I know another way,” Gary said.

“How would you do it?” I asked.

“I’d fold it like this,” he said, folding the square he had made on the diagonal into a triangle.

“How does that prove it’s a square?” I asked.

“See, because the sides match,” he said.

I then took a sheet of paper and folded it into a rhombus. I explained to the class what I did.

“First, I’m finding the midpoint of each side,” I said. I folded each side in half and crimped it to mark its center. Then I folded each corner toward the middle of the paper, explaining that I made the midpoints the end points of each fold.

“Is this shape a square?” I asked.

“No!” A chorus of children answered.

“But look,” I said. “If I measure the sides as Jamie did, they’re all the same.” I measured the sides for the class.

“And if I fold it to make a triangle as Gary did,” I said, “the sides match.” I did this for the children.

“So why isn’t this a square?” I asked.

“The corners are wrong,” Ann said.

“They have to be square,” Jason added.

“Yes,” I said, “measuring the sides is important but it’s not enough to prove a shape is a square. The corners, which we call angles, are also important.”

I demonstrated for the children how to compare a corner of a sheet of paper with the corners of the folded shape to determine whether it’s a square. Then I held up one of each of the three paper shapes I had cut out and introduced the rest of the problem to the class.

“You’re to work in pairs and investigate how to fold the largest square from these three shapes,” I said. “I’d like you to describe in writing how you made each square—including the one you just made from the rectangle, which we’ve already discussed.”

The children’s methods and explanations differed. Some included drawings with their descriptions.

The activity also creates a reason for students to use mathematical terminology and therefore become more familiar with the language of geometry. Writing gives students the chance to describe their experiences with the shapes.
Marina wrote step-by-step directions for folding the triangle into a square.

Jason found it too “hard to describe” how he had folded the triangle into a square and instead drew pictures.

This lesson is valuable for both younger and older students. From tinkering with the shapes, students have the opportunity to learn concretely what a square corner is, that diagonals divide a square into triangles, and how to prove that a shape is a square.
The lesson gives students experience with collecting, organizing, and interpreting data and making numerical inferences from a sample. Carolyn Felux taught this lesson to third graders in San Antonio, Texas.

To begin the lesson, Carolyn explained to the students that they were going to conduct an experiment.

“First, we’ll collect data about how many children in our class are right-handed and how many are left-handed,” she said. “Then you’ll use that information to estimate how many right-handed and left-handed students there are altogether in all the third grade classes in our school.”

Carolyn told the children that each, in turn, would report whether he or she was right-handed or left-handed. She organized the children into pairs. “You and your partner are to record the information as it’s reported,” Carolyn told them. “Talk together about how you’ll collect the information.” She gave the students time to organize their papers.

There were 3 left-handed and 25 right-handed students in the class.

“What information do you now need to estimate how many left-handed and right-handed third graders there are in all the classes together?” Carolyn asked.
The children wanted to know how many third grade rooms there were. Students volunteered the other teachers’ names, and Carolyn listed them on the board. There were six other third grade classes.

The students seemed to feel that this was all the information they needed to make their estimates, so Carolyn gave them further directions. She said, “You and your partner should discuss and agree on an estimate.” She wrote the word *hypothesis* on the board and continued with the directions. “Record your estimate and explain how you decided on it. Your explanation is your hypothesis. It should tell how you used the data to make your estimate.”

**The Children’s Work**

The children’s estimates differed, ranging from 14 to 20 left-handers. Mark and Araceli based their estimate on the data that had been collected in their class and hypothesized that every class in the school (except one) had 3 left-handed students. They wrote: *We think there are 17 left handed and 136 right handed. We think this because we think there are 3 left handed people in every class but Mrs. Duncan we think she has only 2 left handers.*

Mark and Araceli based their hypothesis on the data collected in their own classroom.

None of the children seemed concerned about the lack of precise information about how many children were in each class. The children later collected the information from the other classes to compare with their estimates.

Louis and Mandy guessed, added, and subtracted, but their method wasn’t clear.

Lettie and Sarah didn’t explain how they reached their hypothesis.
Mitsumasa Anno’s imaginative and thought-provoking books delight children and adults. He collaborated with Akihiro Nozaki to create *Anno’s Magic Hat Tricks*. The tricks are logic problems that require students to make conjectures, test their validity, and draw conclusions. Although the initial warm-up problems in the book are simple, they should be read aloud slowly and carefully, as they prepare the reader for harder tricks. The last problem in the book will challenge older students—and even adults. The following describes my work with fourth graders in Mill Valley, California.

**Materials**

*—Anno’s Magic Hat Tricks* by Akihiro Nozaki and Mitsumasa Anno (See the Bibliography on page 179.)

*Anno’s Magic Hat Tricks* has four characters: a hatter, Tom, Hannah, and the reader. The book also has five props: three red hats and two white hats. The author gives the reader the name Shadowchild. Each trick asks the reader to figure out what color hat is on Shadowchild’s head. Shadowchild appears throughout the book as a shadow, making it impossible to see what color hat he or she is wearing.

As I read each problem to the fourth graders, I asked them to offer solutions and explain their thinking. I kept the emphasis on the students clarifying their ideas and convincing one another. They found the early problems too easy.

In the first trick, the hatter has two hats—one red and one white. The illustration shows Tom wearing a red hat. The reader is asked to guess the color of Shadowchild’s hat.
“Maybe this book is better for third graders,” Noah said.
“I’ve read the book to third graders,” I responded, “but it became too difficult for them later on. I wonder if the tricks will get too difficult for you as well.” The children were curious to find out.

In each riddle, Tom can’t see what color hat he’s wearing, but he can see what color Hannah and Shadowchild are wearing. In the same way, Hannah and Shadowchild can’t see their own hats but can see what color hats the other two characters are wearing.

In one riddle, the hatter uses three hats—two red and one white. The illustration shows Tom with a red hat and Hannah with a white hat. When Hannah is asked what color hat she’s wearing, she knows and answers, “It’s white.” The fourth graders easily deduced that Hannah had to see red hats on Tom and Shadowchild in order to know that hers was white. Therefore, they knew that Shadowchild had to be wearing a red hat.

The problems in the book become more difficult when the hatter draws from three red hats and two white hats. The final trick shows Tom and Hannah each wearing a red hat. When Tom is asked the color of the hat he’s wearing, he looks at Hannah’s and Shadowchild’s hats and responds, “I don’t know.” When Hannah is asked, she uses Tom’s response and answers, “Mine is red.” From that information, it’s possible for the children conclusively to deduce the color of Shadowchild’s hat.

After reading this last problem, I asked the children to put their solutions in writing. Sometimes it’s difficult for me to respond to children’s reasoning processes during class discussions when they present many different ideas. When children commit their thinking to writing, I’m better able to learn what each child understands. I can return to their papers later and reflect on their ideas. I can use what I learn from their individual points of view to prepare follow-up lessons.

The Children’s Writing

While some students were unable to figure out a solution or explain their thinking, others were able to reason to a conclusive solution. Jacob, for example, wrote: I think that Shadow child’s hat is white because Tom doesn’t know because Shadow child’s hat is white. Hanna’s hat is red. Hanna knows the Shadow child’s hat is white and tom’s is red. She knows that tom doesn’t know. So hers must be red.

Jacob explained his reasons for believing that, in the last problem of the book, Shadowchild was wearing a white hat.

I think that Shadow child’s hat is white because Tom doesn’t know because Shadowchild’s hat is white. Hanna’s hat is red. Hanna knows the Shadow child’s hat is white and tom’s is red. She knows that tom doesn’t know. So hers must be red.

Peter wrote a run-on sentence that said it all: I think that sadow child’s hat is white because if Tom said “I don’t know” there must be at least one red hat on either Hannah or sadow child and if she saw shadow child with a white hat he would know hers was red and she did so shadow child must have a white hat on.

Some children reasoned differently. Lauren, for example, wrote: I think that shadow child’s hat is white. Because, we started out with 3 red and 2 white. Then Tom and Hana are wering red so now we have 1 red and 2 white. So there for, there is more chances for me to have a white hat.
A Class Discussion

The next day, I returned to the problems, first returning to the simpler tricks. We acted them out, with students volunteering to be Tom, Hannah, and Shadowchild. This helped more of the children see how to use the information in the clues.

Responses from Other Students

I asked the children for a thumbs-up or thumbs-down review of the activity. Seven of the children were intrigued and wanted more. Most of the class thought the beginning problems were okay but felt that the others were better for older children. A few children said they weren’t interested.

“There are many different areas of mathematics,” I reminded them, “and some people find some more interesting than others.” I told them that logic was just one of the different strands they would be studying.

I think that Shadowchild’s hat is white because Hannah said mine is red so the only way she could tell is because there are two white so if you were wearing a white then she would a red.
Every year, Dee Uyeda, a third grade teacher in Mill Valley, California, incorporates into her curriculum the study of haiku, an ancient Japanese form of verse that is composed of three non-rhyming lines. Traditionally, as written in Japanese, the first line has five syllables, the second has seven, and the third has five. Dee does not impose this structure on the children's haiku. She has them concentrate on their thoughts rather than on the number of syllables.

The following paragraph is an excerpt from a paper that Dee wrote:

"Young children, like almost everyone else, benefit from studying haiku. By reading and hearing haiku, thinking about it, and learning to write it, children learn to observe their world more closely and with greater care. They learn to trust their own perceptions since haiku records a personal, individual, and immediate experience—and no one else can see or hear or taste or think about or respond to something quite the way someone else can. Haiku is a way of communicating that personal experience to another person."

Dee became curious about how children might combine their thinking about math with writing haiku, about what they would choose to capture about mathematics. The following are some of the poems the children wrote.
Dee’s third graders write haiku about mathematics.

```
Math
Complicated
Then easy.

5 jelly beans 4 people
So if I eat 1
It will be fair.

Calculator
Can’t add or subtract
Without me.

Math
Just patterns
Waiting to be found

Triangle
You’re half a square
But just as good.

Division
A copycat
Of multiplication
```

Some of the third graders illustrate their haiku.

```
5

12

Tick Tock
GOES the clock
AS I WAIT

Zero
Is as important
AS nine.

Numbers
Come alive
In your fingers.
```
Bonnie Tank developed this lesson to present students with a problem-solving situation that called for division. Rather than focusing on teaching an algorithm, Bonnie had the students invent their own ways to divide. In this way, they had the opportunity to create procedures that made sense to them. Bonnie taught this lesson to fourth and fifth graders in Piedmont, California.

**Materials**
- One jar filled with dry beans (Note: Be sure that you know how many beans you put in the jar.)
- One coffee scoop

Bonnie showed the students the coffee scoop and the jar filled with beans.

“How many scoops of beans do you estimate are in the jar?” she asked.

After the students made estimates, Bonnie emptied the beans from the jar and put five scoops back in.

“Using this information,” she asked, “what estimate would you now make about how many scoops would fill the jar?”

Bonnie asked the students to discuss their estimates with their partners. Then she had some students report their estimates and explain their reasoning as well.

Bonnie added more scoops of beans to the jar. As she did this, some students revised their
estimates. There were both cheers and groans from the class when they learned it took 12 scoops to fill the jar.

“I knew it would take 12 scoops,” Bonnie said, “because I filled the jar last night. I also know there are 334 beans in the jar because I counted them.”

Bonnie then posed a problem. “Work with your partner,” she said, “and find out how many beans you think would fill the scoop. Report the answer you get and also how you figured it out.”

The Students’ Work

The students approached the problem in different ways. Several pairs used multiplication with trial and error. Their papers showed a lot of figuring. Robbie and Christine, for example, wrote: We multiplied numbers with 12 and if the answer was more than 334 the number should be lower. 336 was the closest we could get to 334. We got 28 beans in each cup.

John and Rachel wrote: We found out how many beans were in each scoop by multiplying 12 (the number of scoops) by what we thought was how many in each scoop. When we tried 27 x 12 we got 324 . . . not quite enough. Then we tried 28 . . . 336. The answer is between 27 and 28.

Anne and April did the same, but reported their answer differently. They wrote: 10 of the cups have 28 in them and 2 of the cups have 27 in them.

Some students drew circles to represent the scoops and either drew beans in each circle or wrote numbers. Adam and Erin, for example, drew 12 circles, wrote the number 10 in each, and found the total. They wrote another 10 in each and found the new total. Then they tried a few other numbers and finally decided on 8. They checked by multiplying.

Adam and Erin grouped and added, then multiplied to check their answer.

Gretchen and David used a different approach. They wrote: We solved this problem by finding out what half of 334 is. Half was equal to 167. Then we divided it by 6 because we didn’t know how to divide by 12. Then we checked it by multiplying 27 x 12 and that was too low. That answer is 324. So we tried 28 x 12 & we got 336. So the answer is between 27 & 28.

Gretchen and David used a combination of multiplication and division to find the number of beans in a scoop.
Two pairs of students solved the problem with the standard division algorithm. Nicole and Kelley divided 334 by 12 and got 27 with a remainder of 10. Their explanation outlined in detail how they divided to get an answer. They wrote: *Your answer is 27 remainder 10 beans can fit in each spoonful to put in the jar.* When Bonnie questioned the girls about filling a scoop with “27 remainder 10 beans,” they amended their answer to: *Around 27.*

Bonnie gave students scoops of beans to verify their solutions. Of course, the number of beans varied, and this resulted in a lively discussion about the differences in sizes of beans and in how full each scoop was.

The emphasis in this lesson was on having students figure out what to do to solve a problem, decide on a method, and arrive at an answer that made sense to them. Problems such as this one can help students understand division and its relationship to multiplication.
I developed this lesson to assess students’ understanding of fractions and learn about their ability to apply fractions in different problem-solving situations. I taught the lesson to fourth and fifth graders in West Babylon, New York, and to fifth graders in Mill Valley, California.

When teaching fractions, I find it’s important to assess regularly what students understand. I gather information about what students are learning in three ways: from leading class discussions about students’ reasoning, from observing and listening to students talk with one another as they work on problems, and from examining the written work that students produce. These types of assessments aren’t perceived by students as “testing” situations. Rather, they’re similar to other lessons in which students grapple with problems, share their solutions, and reflect on their thinking.

A Class Discussion

To initiate a class discussion about fractions, I wrote three statements on the board:

1. When pitching, Joe struck out 7 of 18 batters.
2. Sally blocked 5 field goals out of 9 attempts.
3. Dick did not collect at 14 of the 35 homes where he delivers papers.

I read the first statement aloud and asked the students whether Joe had struck out exactly half, about half, less than half, or more than half of the batters. Below the statements, I wrote:

- Exactly half
- About half
- Less than (<) half
- More than (>) half

I invited all of the students who were interested to express their ideas. Most thought that “less than half” made sense.

Philip said, “It’s less than half because 7 and 7 is 14, and that’s less than 18.”

Sophia said, “I think the same, but I used multiplication and did 7 times 2. That’s 14, and it’s smaller than 18.”

Mira said, “Nine is half of 18, and 7 isn’t enough.”

Emmy thought that “about half” was a better estimate. She was the only student who thought this. “It’s pretty close,” she said, “because it’s just 2 off from being exactly half.”

Nick thought that “less than half” made better sense. “I did it with subtraction,” he explained. “I did 7 take away 18, and that’s 11 and that’s more.” I didn’t correct Nick’s explanation of the subtraction.

Several other students also expressed ideas that were similar to those already offered, so I moved on to the next statement: Sally blocked 5 field goals out of 9 attempts.

“It’s more than half,” Angelica said, “because it would have to be 10 to be exactly half.”

“I think it’s just about half,” Josh said, “because 5 is about half of 9.”

Chris was thinking about the situation numerically. He said, “You can’t take half of 9 because it’s an odd number.

“Yes, you can,” Raquel responded. “You can take half of any number. The answer is just in the middle.”

“What is half of 9?” I asked.

“It’s 4½,” Eli said. “I agree with Raquel that you can always take half.”

“But you can’t count half of 9 goals,” Daniel argued. “That doesn’t make any sense.”

“Well, you should be able to take half of anything,” Raquel said.

I offered my perspective. “Can you divide 9 apples in half?” I asked. Most of the students nodded.

“Half of 9 apples is 4½ apples,” Ali said.

“What about 9 balloons?” I asked. The students laughed.

“You’d wreck a balloon,” Eli said.

“It would be stupid,” Sarah added.

“There are things we can divide in half and other things that we can’t,” I said. “When we study fractions, some things that make sense with numbers don’t always make sense when you think about them in a real-life situation. It’s important that we pay attention to how we use fractions as we learn about them.”

Raquel’s next comment shifted the direction of the discussion. “I have something to say about the first sentence, about Joe the pitcher,” she said. “I think it’s closer to 1⁄3 than 1⁄2.”

“Why do you think that?” I asked.

Raquel explained, “Because you can divide 18 into thirds—6 plus 6 plus 6. And 7 is just one away from 18. But it’s 2 away from 9, which is half. So it’s closer to 1⁄3 than 1⁄2.”

“Maybe I need to add another choice to the list,” I said. I wrote About one-third on the board.

I then went on to the third statement: Dick did not collect at 14 of the 35 homes where he delivers papers. It seemed clear to most of the students that “less than half” was the best descriptor for that statement. The students’ explanations were similar to the ones they had given for the other two statements.

Raquel surprised me with her observation.

“I think that 14 is 2⁄5 of 35,” she said. “Look, two 7s make 14, and three 7s make 21, and 14 plus 21 is 35, so 14 is 2⁄5.”

I thought for a moment to understand Raquel’s reasoning. None of the other students, however, seemed interested.

“I agree, Raquel,” I said, to acknowledge her contribution, but I didn’t pursue it further.

I have discussions such as this one frequently. I learn what some of the students are thinking,
and the children have the opportunity to hear other points of view about reasoning with fractions.

At this time, I ended the discussion and introduced a fraction activity.

A Written Assessment

After a similar discussion with another class, I gave the students two written problems I had prepared for them to solve in groups. I asked the groups to find a solution to each problem and explain their reasoning. Also, I asked them to indicate if the problem was “Too hard,” “OK,” or “Too easy.”

Problem 1. At her birthday party, Janie blew out 3⁄4 of the candles. Draw a picture of the birthday cake and candles, showing which candles were blown out and which are still lit. P.S. How old is Janie?

After the students finished their papers, we discussed Janie’s age. There were some giggles when the students realized that groups had arrived at three different answers to the question of Janie’s age—12, 15, and 16.

Seeing the drawings with 12 candles and 16 candles convinced some that both answers made sense. The explanations offered by some of the students seemed to convince the others. The group that had said Janie was 15 had correctly drawn a cake with 16 candles and 4 left lit. They justified their answer by explaining that on birthday cakes there’s an extra candle for good luck, and that’s why they thought she was 15.

I asked the students if Janie could be 4 years old. They agreed that this was possible.

“Could she be 40?” I then asked. This sparked some students to suggest other ages she could be. “Could she be 212 years old?” I asked. Some started to figure in their heads until Kurt responded that she’d be dead and wouldn’t have a birthday cake. The others groaned.

One group reported that they’d had a dispute about the problem. Jason thought Janie could be 12 with 3 candles left burning; Erica insisted she had to be 16 with 4 candles burning. Using either 3 or 4 seemed to be important to the children because those numbers appeared in the problem’s fraction. Erica explained how they resolved the problem. “I just took the paper and recorded my answer,” she said, “and Jason said it was okay.”

Problem 2. Three people are to share two small pepperoni pizzas equally. Shade in one person’s share. Describe how much pizza each person gets.

Groups had different solutions to this problem. All four solutions shown on the next page are correct. Notice that the two groups reporting 3⁄6 shaded in the pizzas differently. The answer of 2⁄6 relies on the interpretation that the two pizzas together make one whole, so each person gets 3⁄6 of the two pizzas. The answer of 5⁄3 pieces is also correct if the pizzas are divided into 8 slices each.

Children benefit from doing many problems in a variety of problem-solving contexts. Even when children have a basic understanding of fractions, they benefit from many opportunities to cement the concepts.
Here are four different ways that students correctly solved problem 3.

3. Three people are to share these two small pepperoni pizzas equally. Shade in one person’s share.

Describe how much pizza one person gets.

2/3 of one pizza.

☐ Too hard  ☐ OK  ☒ Too easy

Another Class Discussion

After we discussed these two problems, I posed another problem for the students to consider. I was interested in seeing how the students would think about adding fractions.

“If I gave Jason 3/4 of a chocolate bar and gave Tara 3/4 of a chocolate bar, how much would I have given away?” I asked. The students offered several answers—6/4, 6/8, 2, 1 1/2. Not many were sure of their answers.

“When I add the 3/4 of a bar I gave Jason and the 3/4 of a bar that I gave Tara, will my answer be greater or less than 1?”

It seemed clear to most of the children that it would be greater than 1. One student explained, “Because you are giving away more than one bar.”

“Would the answer be greater or less than 2?” I asked. One student said that it had to be less because each got less than one chocolate bar.

But when I pushed the students to figure exactly how much chocolate I had given, they found it difficult. Eric said that he didn’t think it could be 6/4 because fractions weren’t supposed to have the big number on top. Erica’s answer was 1 1/2, but she couldn’t explain how she got it.

Finally, I drew two chocolate bars on the board, divided each into fourths, and shaded in the two shares. That seemed to help. These kinds of examples help students form their own mental models for working with fractions in real-life situations.
Learning fractions is difficult for children. When teaching fractions, I engage students with games, problem-solving activities, and explorations with manipulative materials, all designed to provide a variety of ways for students to think about fractions. Recently, in my work with fifth graders in Mill Valley, California, I taught a game to help students learn to compare fractions and use fractional notation. After students had a chance to play the game during a few class periods, I used the context of the game to assess their understanding.

To give the fifth graders experience with comparing fractions, I introduced a game that involved both luck and strategy. I drew on the chalkboard:

“This is a game for two people,” I said. Hands flew up from students who were interested in volunteering to play. I drew another game board and invited Sarah to join me at the chalkboard.

I explained the rules. “To play this game, we’ll use a regular die. We each roll the die four times, taking turns,” I said. “Each time I roll, I write the number that comes up in one of my boxes. And you do the same for the numbers you roll.”

Materials

- Dice, one die per pair of students
- 1–9 spinner, one per pair of students
I stopped to write on the chalkboard:

1. Each player rolls four times and writes the numbers.

I continued with the directions. “Once you write a number in a box, you can’t move it. When we’re done, we’ll each have a fraction and two extra numbers.”

“What are the two extras for?” Jeff asked.

“They’re reject boxes,” I answered. “They’re places where you can write numbers that you don’t want to use for your fraction.”

“How do you win?” Paul asked.

I answered, “When you’ve each written your numbers, you compare your fractions. The larger fraction wins.”

There were no more questions, so I said to Sarah, “Would you like to go first or second?”

“I’ll go first,” she said. She rolled the die and a 2 came up. Sarah hesitated, thinking about where to write it.

Other students called out suggestions. “Put it on top.” “No, stick it in one of the extra boxes.” “I’d put it on the bottom.” “Yeah, the bottom is much better.”

Sarah wasn’t sure what to do but finally decided to write the 2 in the numerator.

Next, I rolled a 3. I put it in a reject box.

Sarah and I continued taking turns. After we each had rolled four numbers, the results were as follows:

<table>
<thead>
<tr>
<th>Sarah</th>
<th>Ms. Burns</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

“You lost, Sarah,” Maria said.

“By a lot,” Seth added. Sarah shrugged, grinning.

Some students weren’t as sure about the outcome. There was a large spread among the students in the range of their understanding about fractions. I continued by giving additional directions for the game.

“Let me explain the next part of the game,” I said. “You each make a drawing to represent your fraction. Because you want to compare your two fractions, you first have to agree about the shapes you’ll use for the whole—circles, rectangles, or something else. That way, you’ll be able to compare fractional parts of the same wholes.”

I turned to Sarah. “What should we draw?” I asked her.

“Circles,” she said.

“You go first,” I said. Sarah drew a circle, divided it into thirds, and shaded in two of the sections.

As I drew my fraction, I talked aloud to explain what I was doing. “My fraction says five halves. If I draw one circle and divide it into halves, that shows two halves. Another circle gives two more halves.” I shaded in all four halves. “I still need one more half.” I drew a third circle and shaded in just half of it.

“You win!” Ali said.

“There’s one more part to playing the game,” I said. “You have to write a math sentence that compares your two fractions. Here are two ways to do this.” I wrote on the board:

\[
\frac{2}{3} < \frac{5}{2} \\
\frac{5}{2} > \frac{2}{3}
\]

I told Sarah that she could return to her seat, and then I completed writing the rules on the chalkboard:
1. Roll four times and write the numbers.
2. Each make a drawing.
3. Write at least one math sentence.

I then said to the students, “I know two other ways to write math sentences that compare these two fractions. Instead of writing \( \frac{5}{2} \), I could count the whole circles I shaded in and then the extra part. That gives me two whole circles and half of another.” I showed the children how to write the mixed number.

\[
\frac{2}{3} < \frac{1}{2}
\]

\[
\frac{1}{2} < \frac{2}{3}
\]

“Who can explain why \( 2 \frac{1}{2} \) is worth the same as \( \frac{5}{2} \)?” I asked. I waited to see who would raise their hands, and then I called on Gwen.

“See, it takes two halves to make a whole circle,” she said, pointing to one of the circles I had drawn. “So another circle is two more, and that’s four, and there’s one more half.” Gwen delivered her explanation confidently.

“I have another way to show it,” Kevin said. He came up to the board, drew five circles, and divided each one in half. Then he shaded in half of each circle and explained how you could put halves together to make wholes.

\[ \frac{2}{3} < \frac{1}{2} \]

\[ \frac{1}{2} < \frac{2}{3} \]

The Next Day

At the beginning of class the next day, Eli made a comment about the game. “I have a strategy for winning,” he said. “Put big numbers on top and little numbers on the bottom.”

“It didn’t always work,” his partner, his partner Joey said, “but it did most of the time.”

“I’m going to give you time to play the game again,” I said, “but this time you’ll use spinners that have the numbers from 1 to 9 on them. That will give you new fractions to compare.”

I had the students play for about 15 minutes before interrupting them for a class discussion. To begin the discussion, I wrote on the board the fractions from one of Tara and Doug’s games.

<table>
<thead>
<tr>
<th>Tara</th>
<th>Doug</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{9}{5} )</td>
<td>( \frac{9}{8} )</td>
</tr>
</tbody>
</table>

“Talk with the person next to you about which fraction is greater,” I said. “Then I’ll have a volunteer come to the board and explain.”

After a few moments, I called the students back to attention and asked for a volunteer. I called on Nick. He came to the board, drew rectangles, and divided them to draw each fraction. “It’s hard to make five parts,” he said, struggling with Tara’s \( \frac{9}{5} \). He erased lines several times until he finally was pleased with what he had drawn. Dividing the other rectangles into eighths was easier for him.

Sophie then came up and showed how she
would do it with circles.

Ali said that she didn’t really need to draw the fractions. “I know that 10 fifths would be two wholes, so that’s just a little less than 2. But 9 eighths is just a little more than 1.”

Some students had difficulties with the fractions. Angie said that she was confused. “I don’t get how to do this,” she complained. Liz admitted that she was confused, too. Jason said, “It’s hard to draw sometimes.”

However, Seth said that he thought these fractions were easy and asked if he could come up and write the sentence. I agreed, and he wrote

\[
\frac{9}{5} < \frac{9}{8}
\]

\[
1 \frac{4}{5} < 1 \frac{1}{8}
\]

two versions:

Over the next several days, I initiated other discussions like this and gave the students practice with drawing and comparing fractions and recording the results with correct fractional symbolism.
The NCTM Standards call for helping students learn to estimate, measure, and “select appropriate units and tools to measure to the degree of accuracy required in a particular situation.” The Standards also suggest that students have experience with measuring inaccessible objects. To engage students with these ideas, Suzy Ronfeldt posed this problem to her fifth graders in Albany, California.

Suzy sent in the work for the Math Solutions newsletter quite a while ago, but I never included it because we couldn’t figure out how to reproduce the large student work. We’ve learned more about computer scanning, so I can now show what the students did.

During the first week of school, Suzy often has her students explore manipulative materials. One year, as students were building trains with Unifix cubes, Suzy heard one child ask another, “Which do you think is longer, the fifth grade hall or the height of the building?” Suzy decided to turn the question into a lesson for the entire class.

The students had various opinions, so Suzy asked for ideas about how they might find the answer. Some children identified the different measuring tools they might use—rulers, yardsticks, meter sticks, and tape measures. Daniel suggested using a trundle wheel.

“It would make it easy to measure the hallway,” he said.

“Yeah, but you couldn’t walk it up the side of the building,” Leonardo retorted.

“I think we’ll need a ladder,” Mariko said.

“What would you do with it?” Suzy asked. Mariko shrugged.

“Oooh, I know,” Amy said. “You could go up to the roof and drop down some string.”

“How do you get up to the roof?” Talia wanted to know. This time, Amy shrugged.
“Maybe we could get the blueprints of the building,” Adam suggested.

Suzy then explained to the students how they would work on the problem. She told them to discuss the problem in small groups and then decide on one way to solve it.

“When you have an idea about what you’d like to do, talk with me about it,” Suzy added. She realized that students would have to leave the classroom in order to do the measuring, and it was important that she know what they were planning to do.

Suzy gave further directions. “I’ll give each group a sheet of 12-by-18-inch paper to report your plan and your results. Include sketches if they’ll help explain what you did. Later, you’ll post your papers so we can compare the results you got and the methods you used.”

The students’ interest was high, and they went to work eagerly.

The Students’ Work

All of the groups decided that the hallway was longer, but their methods of measuring varied. One group estimated the height of the school building by using the height of the steps in the school staircases. They wrote: First we used a meter stick and found out that each step in the staircase is 16 cm high. There are 28 steps in the staircase. They figured that the building was 7 meters and 57 centimeters. Then they used the trundle wheel to measure the length of the hallway and reported it was 28 meters long. They concluded: So the length of the hall is about 3 times longer than the height of the building.

One group figured out a way to use rope to measure the height of the building by doing it in segments. They wrote: What we did was got a wood chunk and tied it on to some rope and threw the wood chunk with the rope on to the cafeteria roof and pulled it down and measured the length of the rope. Then we came back up and came out to the balcony of the school and threw the wood chunk up to the top of the roof from the balcony then we measured that and dropped it down to the cafeteria roof off the balcony and measured that, then we added all the measurements up and got the total of the building which was 30’ 10”. The length of the hall is 92’ 0”. The hall is longer than the height of the building is.

Another group solved the problem in a different way. They wrote: We went down into the office and there was a picture of Cornell School. We measured the picture. On the building we got 113/4”. On the height of the building we got 4 1/2. So this proves that the hall is longer than the height of the building.

The variety of methods and units the students used led to an active class discussion that helped reinforce the idea that there are many ways to solve a problem. The activity was also valuable because it provided an opportunity for the children to connect their mathematics learning with their environment. Students need many concrete experiences with measurement throughout the elementary grades.

This group was one of three groups in the class that used the blocks on the outside of the building to measure and compare the two lengths. They reported their answer in inches.

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**Which is longer—the fifth grade hall or the height of the school building?**

**Hypothesis:** We think the hall is longer.

By the fifth grade hall we mean from the doors of the administration office to the end of room 143. We went outside and looked at the front of the building. We measured the building using the blocks in which the letters of CORNELL are placed in the front of the school. It takes three blocks to go up the school and seven to go across the fifth grade hall. So the fifth grade hall is 333 inches across the hall, so the fifth grade hall is 333 inches longer.

Cornell:

1. 7 blocks across the fifth grade hall
2. 3 blocks up the school building

ELL:

1. 111 inches across the fifth grade hall
2. 777 inches across the fifth grade hall
Although this group also used the blocks on the building, they reported their answer in yards.

This group found a way to approximate the distances by using their hands. Their description of what they did begins on the right-hand panel and concludes on the left.
When preparing for a demonstration lesson, Bonnie Tank was looking for a logical reasoning problem to involve fifth graders. In rummaging through my book *Math for Smarty Pants*, I rediscovered “A Mathematical Tug-of-War.” The problem seemed appropriate because students are familiar with tug-of-war contests (the context for the problem), and there are many ways to find a solution to the problem. Bonnie presented it to fifth graders in Piedmont, California.

Bonnie began the lesson with a general explanation of what the students would be doing. “I’m going to tell you what happened in the first two rounds of a tug-of-war contest,” she told the fifth graders. “Then you’ll use the information to figure out what happened in the third round.” Bonnie read the story to the class:

> Your job in this mathematical contest is to decide who will win the final tug-of-war. The first two rounds give you the information you need.

The First Round.
On one side there are four acrobats who have come down to the ground during the off-season for this special event. They have well-developed arm muscles because of all the

swinging they do, and have proven themselves to be of equal strength. Remember that fact.

On the other side are five neighborhood grandmas, a tugging team that has practiced together for many, many years. They, too, are all equal in strength. Remember that fact also.

In the contest between these two teams, the result is dead even. Neither team can out-tug the other. Remember that too.

The Second Round.
One team is Ivan, the specially trained dog that got his start as a pup when he was taken out for a walk by his owner. Ivan gets pitted against a team made up of two of the grandmas and one acrobat.

Again, it’s a draw—an equal pull. Remember that fact.

It’s the final tug that you must figure out. It will be between these two teams: Ivan and three of the grandmas on one side, the four acrobats on the other. Can you figure out who will win this tug of war?

One way to solve this problem is to use algebra, a branch of mathematics that uses equations to deal with relationships between quantities. If you haven’t learned about algebra yet, you’ll have to rely on logical reasoning. Either way it’s mathematical thinking you must do. Get a pencil and paper to help you tug on this problem.

Bonnie then asked, “Before you figure out what happened in the third round, would you like to hear the story again?” The students were unanimous in their response. Although they had listened attentively, Bonnie knew that they probably didn’t remember the information necessary for solving the problem.

“This time,” Bonnie said, “you might want to take notes.” She gave the students time to gather pencils and paper and then read the story again, pausing at times for the students to make their notes.

The Students’ Work
The students worked in groups of two or three to solve the problem. The consensus was that Ivan and the grandmas would win the third round. However, the ways they reached that conclusion varied greatly.

David and John used a method of substitution. They wrote: \[ \text{Ivan} = 2 \text{ grandmas} + 1 \text{ acrobat.} \] There are already 3 grandmas. Ivan, who is helping them, equals 2 grandmas and 1 acrobat. If you take away Ivan and replace him with 2 grandmas and one acrobat, it will be like round 1 except the grandmas have a acrobat.

Keith and Andy created “strength points” to arrive at a solution: Since there are 4 acrobats, out of 100% each of them has 25 strength points. The grandmas had to be equal so each has 20 strength points because there are 5 of them. In the second round, the teams were; 2 G’s and 1 A against Ivan. To be equal, Ivan had to have 65 strength points. In the 3rd round, the teams were Ivan and 3 G’s against 4 A’s. Adding Ivan’s and the G’s strength points, Ivan and th G’s win!
Keith and Andy assigned “strength points” to the different characters to determine who would win the third round.

Keith and Andy assigned “strength points” to the different characters to determine who would win the third round.

Kari and Caitlin used fractions. They wrote:

Now, say the grandmothers equal 1 unit of work. Each acrobat is stronger than each grandma, so they equal 1 ¼ units of work each. Ivan the dog is equal to 2 grandmas and one acrobat. Their units of work equal 3 ¼ units, so Ivan equals 3 ¼ units of work. So Ivan and 3 grandmas equal 6 ¼ units and the acrobats only equal 5. So Ivan and 3 grandmas win!!!!

Chris, Benny, and Julia also used fractions, but in a different way: 4 acrobats = 5 grandmas which mean that if an acrobat equaled 1 a grandma would equal ⅖. Ivan can take on 2 grandmas and one acrobat. Together that equals 2⅙. 2⅙ (Ivan) + 2⅙ (grandmas) = 5. 4 acrobats together equal 4. That makes Ivan’s team stronger.

Paul and Amelie reasoned with a minimum of numerical calculations. They wrote: Iven can take on a little more than two acrobats. Three grandmas can take on at least two acrobats. Add a little more than two acrobats and two acrobats, you get more than four acrobats. That means Iven’s team has more strength!!

Having the students present their solutions helped reinforce the idea that a problem can be solved in different ways.
This probability lesson gives students experience with making predictions based on a sample and also engages them in thinking about fractions. I taught the lesson to fifth graders in San Francisco, California.

**Tiles in the Bag**

<table>
<thead>
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<th>Grade</th>
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<td>6</td>
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**Materials**

- Blue, red, and yellow Color Tiles
- One lunch bag

To introduce the lesson, I had the students draw samples from a bag into which I had put 5 blue, 2 red, and 1 yellow tile. I told the children only that I had put 8 Color Tiles in the bag and that they were blue, red, and yellow. I drew three columns on the board, labeling one for each color.
I explained to the children that they would draw samples from the bag, and I would record the colors on the board. Many of the children eagerly volunteered to draw samples. I went around the class having each student draw a tile and then replace it in the bag. There was, as there always is with this activity, much cheering. The children cheered for whatever color was chosen. (It seemed that they were more interested in cheering than in what they were cheering for.)

There were 31 children in class that day, and I also drew a tile. The result of the 32 draws was:

<table>
<thead>
<tr>
<th>Blue</th>
<th>Red</th>
<th>Yellow</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

I asked the children to discuss the results in their groups and answer the following questions:

- Predict how many tiles of each color are in the bag.
- What fraction of the draws are blue?
- What is a more commonplace fraction than \( \frac{23}{32} \) that tells about what fraction of the draws were blue?
- Are more or less than \( \frac{3}{4} \) of the samples blue?

**The Children’s Thinking**

Three groups predicted 4 blues, 2 reds, and 1 yellow; two groups predicted 5 blues, 2 reds, and 1 yellow; one group predicted 6 blues, 1 red, and 1 yellow; and one group predicted 4 blues, 2 reds, and 2 yellows. I asked the students to explain their reasoning.

Alex explained, “Since there are 8 tiles in the bag, then four tally marks stand for each tile. Then 23 blues is close to 24, so that would be 6 tiles, and 5 reds is close to 4, which should be 1 tile, and 4 yellows is 1 tile.”

Jerry said, “We think most are blue, but there are more red tiles than yellow tiles, and that’s why we said 5, 2, 1.”

Joyce said, “We thought 4, 2, and 2 because we think there are the same number of reds as yellows, and that there could be more than one of each.”

I took this opportunity to talk with the class about some probability ideas. First I talked about how sampling is used in real life, offering the situation of how polls are made. David added an example. “It’s like when they say that three out of four dentists recommend a toothpaste,” he said. Then I talked about sample size. “Imagine if we had 100 samples instead of 32,” I said, “or 800, or even more. The larger your sample, the more reliable the information. It’s important to mathematicians to have sample sizes large enough to be useful for making predictions.”

I also talked about the students’ predictions. “Not everyone can be right,” I said. “That’s not what’s important. What’s important is how you reason to make a prediction. I’m interested in your thinking because I know that, without looking in the bag, there’s no way to be absolutely sure how many tiles of each color are inside.”

Then I told the class that only one group’s prediction was correct. The students groaned—but not very loudly.
Materials

- One sheet of chart paper

To begin this lesson, I asked the students, “Would you rather be the age you are now, younger, or older?”

After an outburst of discussion, I called the students to attention. On a sheet of chart paper, I set up a way for them to record their preferences and had each student put an X in the appropriate place.

What age would you rather be?
Mark with an X.

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<tr>
<th></th>
<th>Younger</th>
<th>The same age</th>
<th>Older</th>
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<td>X</td>
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In previous lessons, I had modeled how to draw conclusions from data on graphs and given the children opportunities to draw their own conclusions. Also, I had given the students questions to answer based on the data in graphs. In this lesson, I gave them a different task.

After the students had recorded on the graph, I said, “For this graph, you’re to work in your groups and write questions that others can answer from the information posted.”

After the students had written their questions, I had groups take turns reading aloud one question at a time. Then I gave the class instructions.

“When you hear a question, see if you wrote the same question or a similar one. Decide if the question can be answered from the information on the graph.” In this way, I encouraged the students to listen to and think about the questions.

This group asked a variety of questions that required comparing the number of Xs in different categories.

1. Which one was picked the most?
2. What fraction of the class picked younger?
3. How many more people picked older than younger?
4. What fraction of the class picked the same age and younger?
5. If you added the “older” with the “younger” how much would you get?
6. How many people picked the same age?

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The second question on both of these papers had no relationship to the meaning of the graph’s data.

1. What fraction of the class picked the same age? (lowest terms)
2. What would you get if you added the same as the older together and then times it by the younger?
3. How many groups of 2 are there on the graph?

To end the lesson, I raised one more question.

“How many more people chose the same age over younger?”

The students’ unanimous response was yes. Solving problems they create themselves helps make students more willing learners. I had the students record and post their questions. Then, for the rest of the lesson, they worked on solving one another’s problems.
From the Ceiling to Algebra

Jason, a third grader in Mill Valley, California, inspired a pre-algebra lesson based on ceiling tiles. (See Math from the Ceiling on page 93.) I later tried a variation with fifth graders in West Babylon, New York.

Materials

- 10-by-10 grids, one per group of students (See the blackline master on page 174.)

Part of the problem I had presented to third graders was to figure out how many holes there were around the edges of 12-inch-square acoustical tiles, each with a pattern of holes in a square array. With fifth graders, I didn’t use a ceiling tile but posed the around-the-edge problem for a 10-by-10 grid.
“I have a problem for you to solve,” I said, holding up the grid. “I call it the ‘Around-the-Edge Problem.’ How many squares do you think there are around the edge of this 10-by-10 grid? You could count them, but I’d like you to think about other ways to figure it out. Talk about this in your group and see if you can agree on an answer. Then write about your thinking.”

Working in groups of four, the students found five ways to arrive at the answer of 36. As they explained their methods to the class, I translated them to numerical representations on the chalkboard:

1. \((8 \times 4) + 4 = 36\)
2. \(10 + 10 + 8 + 8 = 36\)
3. \((10 \times 4) - 4 = 36\)
4. \(10 + 9 + 9 + 8 = 36\)
5. \(9 \times 4 = 36\)
6. \(100 - 64 = 36\)

(The last method was my contribution; I subtracted the number of squares inside the border from the total number of squares in the grid.)

We then tested to see if the methods would work on other grids—a 5-by-5, a 4-by-4, and a 3-by-3. It was easy for the students to see that they did.

“I’m going to write a formula to describe one method,” I told the class. “I’ll use \(H\) to stand for the number of holes on a side and \(E\) to stand for the number of holes around the edge. See if you can tell which method it is.” I wrote on the board:

\[H + H + (H - 2) + (H - 2) = E\]

It seemed obvious to the students that the formula I wrote described the second method. I then gave the class another problem.

“Your job is to write a formula for each of the other methods,” I said. “When you’ve done this, check that your formulas work for all the grid sizes we tested.”

The groups worked for about half an hour. The method that was most difficult for them was the last one—the one I had contributed. The students posted and compared their formulas, which led to a discussion of different algebraic notations. The activity is rich because it shows that there is more than one way to arrive at a solution. Also, it can be checked concretely by counting around the edge, yet it can also be described abstractly with algebraic formulas.

These students wrote an algebraic formula for each method of solving the around-the-edge problem.
The December 1992 issue of The Oregon Mathematics Teacher (TOMT) included a problem submitted by Jim O’Keefe from Ontario, Canada. The problem was identified as suitable for grades 8 through 12 and described as one that could be solved “by using a combination of standard methods and a little mathematical intuition.” Jan DeLacy thought the problem would be an appropriate challenge for the sixth graders she was teaching in Bellevue, Washington. Cheryl Rectanus later tried it with her fifth graders in Portland, Oregon.

The Problem: A bird collector wants to buy 100 budgies and wants to spend exactly $100. Blue budgies cost $10 each, green budgies cost $3, and yellow budgies cost 50 cents. The collector wants to purchase at least one budgie of each color. How many blue, green, and yellow budgies could he buy?

The article in TOMT included an algebraic solution (let $x =$ the number of blue budgies, etc.), and although Jan knew the sixth graders didn’t have algebraic skills, she felt they could find other ways to tackle the problem.

“It turned out to be one of the best group/partner interactions I’ve seen,” Jan wrote. “I was thrilled at how they attacked the problem. Only a few hung back saying it was too hard and they didn’t know what to do, and as others around them began to try things, even those students got involved. Most used calculators, some used cubes, and some used diagrams and tally marks.”

Jan found the students’ logic interesting, even when they didn’t reach a solution. “I was
particularly fascinated by Dorel’s approach,” she reported. “He found the cost of one budgie of each color—$10 + $3 + $0.50 = $13.50—and used the equals key on his calculator to add $13.50 repeatedly until he reached $108. Then he started subtracting birds.” Dorel managed to spend $100, but he bought only 21 birds.

This sixth grader’s strategy was to investigate possibilities for different numbers of blue budgies.

<table>
<thead>
<tr>
<th># Blue</th>
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<th>Budgie</th>
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<td>10</td>
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<td>$70</td>
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<td>$80</td>
<td>1</td>
<td>$90</td>
<td>1</td>
<td>$90</td>
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</tbody>
</table>

The Lesson with Fifth Graders

When Cheryl Rectanus gave the problem to her fifth graders, she reported, “Some groups used green, blue, and yellow Pattern Blocks to represent the birds, along with play money. Others just used play money but found it wasn’t very helpful. Almost every group used calculators and paper and pencil.”

After the students had worked on the problem for half of a period, Cheryl interrupted them and asked them to discuss what they knew so far. Several groups discovered that a solution would have to include an even number of yellow budgies. However, Cheryl noted that the students seemed able to pay attention either to buying 100 birds or to spending exactly $100, but not to both.

None of the students reached a solution that first period, but Cheryl had them write about the approaches they had used and the combinations that didn’t work.

These fifth graders reported what did and didn’t work.

That night, Cheryl organized their information on a database and sorted it in four different ways—by total money spent, by the total number of birds bought, by the number of yellow birds bought, and by the number of green birds bought. She presented the results the next day, and the class looked for trends, noticing which combination of birds came close to a solution.

“Having the information was interesting to the students,” Cheryl reported, “but it didn’t seem to help them find a solution.”

After 45 minutes of work on the second day, one group found a solution. This seemed to motivate the others to continue their search. On the third day, a second group discovered the same solution as the first group. In a class discussion, Cheryl had the students report their approaches and talk about what they learned and how they worked together.

One group wrote: We approached this problem by playing with the numbers and trying out different combinations. We learned that this problem is a lot easier said than done!

From another group: We learned that this problem requires hard thinking, for instance, we had 99 birds and 99 dollars, and we couldn’t figure out how to make 100 and 100.
Another group wrote: We approached this by taking guesses and finding out if it is right or wrong. We learned that guessing can get you close to the answer.

Cheryl also asked the students for their opinions about the appropriate grade level for the problem. There was a variety of responses. One group wrote: We think that the appropriate grade level for this is grade 5 and up because it’s hard. We don’t think younger kids would get it.

Another wrote: We think it’s 6 grade work because not a lot of people in my class got a 100 birds and a 100 dollars.
At the 1989 NCTM annual conference, Julia Szendrei, a math educator from Budapest, Hungary, presented a session on probability. Kris Acquarelli used the ideas with three sixth grade classes in Poway, California.

Materials
- Color Tiles
- Lunch bags
- List of statements (See the blackline master on page 175.)

Kris put 8 Color Tiles in a bag—4 red, 3 green, 1 blue. She listed the contents of the bag on the board and gave each student a sheet with the following statements on it:

1. All 3 tiles are the same color.
2. All the tiles are red.
3. There is a red tile among them.
4. Not all tiles are the same color.
5. There are 2 red tiles among them.
6. Only 1 tile is red; the other 2 tiles are a different color.
7. There is no blue tile among them.
8. There is a green tile among them.
9. There is 1 blue tile.

Probability Tile Games
Kris had each student draw a seven-step game board.

START ○○○○○○○○○ FINISH

Then she gave the instructions for the first game:

**Game 1.** Choose one statement. Students in your group take turns drawing a sample of 3 tiles from the bag and revealing the colors to the group. With each draw, you evaluate whether the statement you chose is true for the 3 tiles. If so, mark off one step on your game board. The idea is to reach the last step as quickly as possible.

Kris had the students play several rounds of the game in groups of four, then discuss which statements provided the best strategies.

“Some students asked me about the wording of the clues,” Kris reported, “and I simply told them to decide in their groups.”

After the students had had a chance to play, Kris asked them to write about which statements they thought were best to choose. She also asked them to write about the statements they felt weren’t clear. The students shared their ideas in a class discussion.

The Students’ Writing

“In all three classes,” Kris wrote, “most students felt that although there were four strong strategies, number 3 was most likely to produce a winner.”

Sam believed that statement 3 was one of two best statements. He wrote: *There isn’t one best statement there is two. They are: There is a red tile among them (#3) or there is a green tile among them (#8).*

Julia had a different opinion. She wrote: *I think the best choice is number 4 “not all tiles have the same color” because it’s very rare that you get all of one color. I think all of the things are clear.*

Later, Kris introduced two other games to the students.

**Game 2.** Pick two statements. After 3 tiles are drawn, exactly one of the statements you chose must be true in order to mark off a step.

**Game 3.** Choose one statement and then write another one that is different from those given. As with game 2, after 3 tiles are drawn, you mark off a step if exactly one of your statements is true.

Again, Kris asked the students to pick the most useful statements and write about why they were best.
Delia picked statements 3 and 7 as best for game 1, statement 3 for game 2, and statements 3, 5, and 8 for game 3. For game 2, she wrote:

*Every time someone played #3 they won because there is 4 reds and 7 because there is only 1 blue.*

Delia found that statement 3 was useful for all three games.

<table>
<thead>
<tr>
<th>the best strategies are Game 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. number 3, 7</td>
</tr>
<tr>
<td>2. Every time someone played #3 they won because there is 4 reds and 7 because there is only 1 blue</td>
</tr>
<tr>
<td>3. # 3, 5 and 8 because on #3 and #5 it says there is a red or green tile among them and you could pick more than one and you did know if there could be more than 1 and #5 says there are 2 red tiles among them, we don't know if there could be more than 2 or just 2.</td>
</tr>
</tbody>
</table>
Investigating pentominoes is a well-known math activity in which students search for pentominoes by arranging square tiles so that each touches the complete side of at least one other. While Kris Acquarelli was on special assignment in Poway, California, she extended the pentomino exploration with Blanche Gunther’s sixth graders. Using interlocking cubes that snap together on all sides, Kris had the students investigate penticubes—three-dimensional shapes made of five cubes each.

Kris distributed the cubes and asked each student to make a shape by snapping together five cubes. “Compare the shapes made in your group,” she said, “and talk about how they’re the same and how they’re different. Ignore the colors, and just pay attention to the shapes of the constructions.”

The students went to work. Kris had worked with Blanche’s class before. She reported, “Students sit together in groups of four, they talk with one another as they work through problems, and they are used to listening to other groups describe their work and thinking.”

Materials

- Multilink or Snap Cubes
- Paper bags (larger than lunch bags), one per group of students
- 2-centimeter grid paper (See the blackline master on page 176.)
- Isometric dot paper, optional (See the blackline master on page 177.)
While the students were working, Kris built three shapes with five cubes each, two that were identical except for the color of the cubes and another that was a mirror image of them.

When Kris called the students back to attention, she asked what they had noticed about their shapes. There was a variety of responses: “Ours were all different.” “Some are in just one layer and some have two or three layers.” “I think there are a lot of different ones.” “Some are simple and some are complicated.”

Kris then showed the class the two identical shapes she had made, holding them in different orientations. “Except for the different colors used,” Kris told the class, “I think these two shapes are the same. Can you see why I say that?”

Several students responded. “That can’t be.” “They don’t look the same.” “Turn that one around, no, the other way.” “Oh, yeah, I see it now.” “I don’t.” “See, they both have an L-shape with one sticking up from the short end.”

Kris held the shapes so that it was easier to see that both were the same. She then showed the mirror-image construction she had made. “How is this configuration like the other two, and how is it different?” she asked.

Again, several students had ideas. “It has the same L.” “Yeah, but the other cube isn’t sticking up the same.” “It would be if you turned it.” “No, that doesn’t work.” “Weird!”

“This shape is a mirror image of the other two,” Kris told the class. “With the extra cube on top, the L-shapes are reversed.”

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**The Penticube Activity**

Kris then introduced the activity. “Work in your groups,” she said, “and see how many different configurations of five cubes you can make. Consider shapes that are mirror images of each other to be different. I’ll leave these three shapes up here for you to examine if you’d like.”

“How many shapes are there?” one student asked.

“I don’t know,” Kris answered honestly. “I learned recently that there’s a commercial set available of these shapes, but I don’t know how many come in the set. That gave me the idea of having you find what shapes were possible.”

Kris’s response satisfied the students, and they were eager to begin. Before they got started, Kris gave a suggestion.

“As you look for shapes,” she said, “be sure to organize your group so you can check for duplicates and eliminate those that are the same.” The students got right to work.

Kris realized that the students could work on this problem for much longer than the class period would allow. However, she interrupted them when about 15 minutes remained to discuss what they had done.

“How many different shapes have you found?” she asked. The range reported by the groups went from 17 to 35.

“Let’s collaborate and see what we know so far,” Kris said. “You’ll be able to continue your search tomorrow.”

Kris then distributed a paper bag to each group. She asked one group to hold up a shape and describe it.

“If your group has this shape,” Kris told the rest of the class, “put it in your bag. If not, make one for your bag.” There was a busy search as groups rummaged for the shape.

After three rounds, Kris realized that this approach had several problems. It was too time-consuming, students couldn’t see one another’s shapes well, and there wasn’t enough individual participation. She could tell that students were losing interest, and she abandoned the plan.

“Just put all your shapes in your bag,” she said, “and you’ll get back to work tomorrow.”
Additional Explorations

After class, Kris and Blanche met and brainstormed further explorations. Kris reported, “We wanted students to experience activities that help visual and spatial perception, to use concrete materials to explore geometric concepts, and to communicate their ideas with one another as they problem solve together.”

Following are the activities Kris and Blanche developed to use the student-made constructions.

- **Build a Wall.** Using a file folder or notebook, one student hides a penticube and describes it for a partner (or for others in the group) to build. Afterward, students discuss which directions were useful and which were confusing or ambiguous.

- **Penticube Jackets.** Using 2-centimeter grid paper, students cut a pattern that can be folded to fit a penticube exactly. (Jackets are nets that fold to cover all faces of the penticube.)

- **More Penticube Jackets.** Students take one penticube and find as many different shaped jackets as possible. (Will the number of possible jackets be the same for all of the penticubes?)

- **Surface Area.** Students compare the surface area of different penticubes and see what they notice about the shapes of penticubes with more and less surface area.

- **Perspective Drawing.** Students practice drawing three-dimensional versions of penticubes. They can use isometric dot paper for these drawings.

- **Building Rectangular Solids.** Students try to put together several identical penticubes or several different penticubes to make rectangular solids. They investigate the sizes and shapes of rectangular solids they can build.

- **Penticube Riddles.** Using 2-centimeter grid paper, students draw top, bottom, and side views of a penticube. They cut out their three drawings, paste them on a tagboard card, and staple the card to a paper bag that holds the actual penticube. Other students try to build the shape using the views drawn. They check their construction by looking in the bag.
In November 1992, David Ott developed a unit on multiplication and division of whole numbers for his sixth graders in San Lorenzo, California. David was interested in having the students solve problems from real-world contexts. The unit consisted of six problems that involved students in arithmetic calculations and required them to gather data, measure, and use geometric ideas.

"The students had a wide range of abilities," David reported. "Some had good instincts and number sense but slow or shaky arithmetic skills. Some had passing arithmetic skills but very little number sense. And while some showed strengths in both number sense and arithmetic, others showed strengths in neither."

Working in groups of four, students were to choose three of the six problems in the unit, solve them, and present two of their solutions to the class. David directed the students to put each solution on a separate sheet of paper and asked them to show all of their work. He also told them that all group members had to understand their group’s work and contribute to group reports.

Following are the directions for four of the problems in David’s unit. (The other two problems related to previous work the students had done.)

- Replacing Floor Tiles. Figure out the number of floor tiles needed to tile the floor of our classroom. Then calculate the number of floor tiles needed for our wing and for all the classrooms in the school.
- **Stacking Chairs.** If we stacked all of our chairs on top of one another, how high would the stack reach?

- **Head to Heels.** Figure out the length of your group if all of you were to lie heel to head. Then use that information to approximate the length of all the students in our class lying heel to head.

- **Filling the Room with Tables.** Figure out the maximum number of tables that could completely fill the classroom floor area.

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**Solving the Problems**

David reported that the problems engaged the students, and all were involved in solving the problems and preparing and presenting reports. Students’ solutions to the problems were quite different. For example, for “Replacing Floor Tiles,” students had to decide how to deal with places where whole tiles weren’t needed. Could some tiles be cut and each part used, or would whole tiles be necessary and the unused portions discarded? In “Stacking Chairs,” students measured the chairs with differing degrees of precision. Also, some groups included the teacher’s chair and some didn’t. Answers varied for “Head to Heels” because each group used the heights of its members to approximate the length of the whole class laid end to end.

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For “Head to Heels,” these students used the heights in their own group to estimate the heights of the rest of the students.

---

\[
\begin{array}{c}
Raul \text{ is } 5’1”, \text{ Jenny is } 4’11”, \text{ Miguel is } 5’4” \text{ and Amanda is } 4’11”. \\
Raul = 5 \times 12 = 60” + 1” = 61” \\
Jenny = 4 \times 12 = 48” + 11” = 59” \\
Miguel = 5 \times 12 = 60” + 4” = 64” \\
Amanda = 4 \times 12 = 48” + 11” = 59”
\end{array}
\]

\[
\begin{array}{c}
\frac{3}{61} \times \frac{3}{3} = \frac{3}{243} \\
\frac{59}{64} \times 7 = 1701 \\
\frac{59}{243} \\
\end{array}
\]

There are 7 table groups, 4 people at each. 28 people in the class. (We evened the number 29 to 28.) The height of all the people at our table is 243”.

The estimated height of the class is 1701” by our height.
In this riddle activity, students combine their number sense and logical reasoning abilities to write mathematical riddles and clues. Cathy Humphreys taught this lesson to sixth graders in San Jose, California.

To introduce the activity, Cathy gave the class a riddle to solve. “I am thinking of a number,” she said. “See if you can figure it out from these clues.” Cathy had written the clues on an overhead transparency. She showed the first two clues to the class.

Clue 1. The number is less than 30.
Clue 2. It is a multiple of 3.

“Talk in your groups about what you know from the clues,” she told the class.

After just a few moments, the students reported the numbers that fit these clues: 3, 6, 9, 12, 15, 18, 21, 24, and 27.

“I know one more number that fits these clues,” Cathy said. The students were stumped, so Cathy reminded them about zero and added it to the list. Then she showed the next clue.

Clue 3. The number is odd.

Groups huddled together and quickly eliminated the even numbers and agreed that the
possibilities now were 3, 9, 15, 21, and 27. Cathy then gave the fourth clue.

Clue 4. The number is not a square number.

“So it can’t be 9,” Jon blurted out. The others agreed. Cathy displayed the last clue.

Clue 5. The sum of the digits is 6.

It was now obvious to the students that the only possible answer was 15.

“I’m going to give you another set of clues,” Cathy then told the class. “This time, I’ll show you all the clues at once. Work in your groups to solve the riddle.”

Clue 1. The number has two digits.
Clue 2. It is a factor of 60.
Clue 3. The number is not a multiple of 10.
Clue 4. The number is not prime.

Cathy’s previous experience had taught her that two errors were typical when students wrote their own riddles: Clues sometimes led to more than one possible answer, and riddles sometimes included redundant clues. To address these possible pitfalls, Cathy purposely included both of them in her second riddle. Both 12 and 15 were possible solutions, and clue 4 was unnecessary.

Two groups arrived at the answer of 12, three got 15, and two groups discovered that both numbers were possible.

“Then my riddle is incomplete,” Cathy said when the class agreed that two answers were plausible.

“Which one is right?” Kelly asked.

“The number I was thinking of was 15,” Cathy responded. “In your groups, think of a clue that would make 15 the only possible answer.”

The groups came up with several suggestions: “The number is odd.” “The number is a multiple of 5.” “The number isn’t even.” “If you add the digits, you get 6.”

Cathy then said, “My riddle was incomplete, but I also included a clue that wasn’t necessary at all.”

“Oh, yeah,” Nathan said, “you didn’t need to say the number wasn’t prime. We already knew that.”

“How did you know?” Cathy asked.

Tami answered. “It couldn’t be 2, 3, or 5 because you said it had to have two digits, and those were the only possible numbers that were prime.”

“A clue that’s unnecessary is called a redundant clue,” Cathy told the class. She wrote the word redundant on the overhead.

Cathy then had the students work in pairs to write their own riddles. “Be sure your clues lead to only one answer,” she cautioned, “and check to make sure you don’t have any redundant clues.”

These students wrote complete riddles without redundant clues.

We are thinking of a number

cue 1: Our number is less than 20.
cue 2: It’s a multiple of 2.
cue 3: It’s not square, it’s not prime.
cue 4: One number has six factors.
cue 5: The number is higher than twelve.

We are thinking of a number

cue 1: Our number is less than 40.
cue 2: Our number is an odd number.
cue 3: Our number is not a square number.
cue 4: Our number is not prime.
cue 5: Our number is not a multiple of 5.
cue 6: Our number is a factor of 168.
Students often first encounter the word *polygon* in a textbook definition usually followed by several pictorial representations. The idea of what a polygon is isn’t complicated, but the word is not commonly used outside of the math classroom, so students often have trouble remembering what it means. Cathy Humphreys developed this lesson to help her students in San Jose, California, learn what polygons are.

**Materials**
- Geoboards, one per student
- Rubber bands

Cathy had planned to use geoboards to teach area. When she asked her students to make a polygon on their geoboards, she was met with blank stares.

“What’s a polygon?” the students wanted to know.

Rather than continue the lesson she had planned, Cathy decided to provide an opportunity for the students to learn about polygons. She gave the class the following direction. “Use just one rubber band and make any shape you’d like on your geoboard.”

Once the students had done this, Cathy asked 12 of them to prop their geoboards on the chalkboard tray, so everyone in the class could see their shapes. She sorted the geoboards into two sets.
“All of the shapes on the left are polygons,” Cathy explained. She continued, “None of the shapes on the right are polygons. Use this information and discuss in your groups what you think a polygon is.”

These shapes are polygons.

These shapes are not polygons.

Students discussed their ideas and presented them to the class. Cathy then had the students whose geoboards were not on the chalkboard tray add them to the correct sets, explaining why they were or were not polygons. Finally, students wrote definitions of the word *polygon* and read them to one another.

**Note:** This lesson, taught to fifth graders, appears on the geoboard videotape in the “Mathematics with Manipulatives” series. (See the Bibliography on page 179.)
My dad was a one-and-a-half-cup-of-coffee man. The cup of coffee he enjoyed after dinner was never quite enough. When offered a refill, he always replied with the same request, “Just half a cup, please.”

Over the years, no matter how much coffee was poured into his cup for that second serving, it earned a comment. “Look at that, the waiter must need glasses!” was one standard. I’m not sure that my dad ever felt that he was properly served half a cup of coffee.

More recently, six of us were having dinner out and a friend asked for half a cup of coffee when offered a refill. The waitress filled her cup. This time it was I who couldn’t resist a comment.

“Do you think your cup is half full now?” I asked. The cup was a standard, rounded cup, much wider at the top than at the bottom.

Everyone at the table looked at me. They peered into the cup, then looked at me again, then peered into the cup again.

“Is this one of your math teacher questions?” someone asked.

A discussion began about where the halfway mark would be on the cup. The discussion became animated and began to get rowdy. It occurred to me that this might be an interesting question to raise with students. The following week, I brought the problem to a class of sixth graders in Mill Valley, California.
**Materials**
- Plastic glasses that are wider at the top than at the bottom, one per pair of students
- Dried beans or rice
- Measuring cup, 1 cup or larger, at least one
- Measuring spoons

For this lesson, I wanted the students to engage in two activities: predict at what level the plastic glass would be half full and design a procedure for finding the halfway mark.

I organized the class into pairs; there were 11 pairs and 1 group of three. I gave each group a plastic glass and explained the activity.

“After you and your partner discuss and mark where you think the halfway point might be,” I said to the class, “I’d like you to design a procedure that you could use to find it. Please write your procedure clearly enough so others can use it.”

Most students eyeballed the glass to predict where the halfway point was. One pair, however, took careful measurements of the diameter of the top, the diameter of the bottom, and the weight of the glass. They felt sure that the halfway mark could be calculated from these measurements. But they were stumped as how to do so, and they finally made a guess.

As students finished their work, I first checked that they had written their procedures clearly. Then I gave them two choices—either use their procedure to find the actual halfway mark or design an alternative procedure. In this way, all students stayed engaged until every pair had completed at least one procedure. This took just about 15 minutes. I then called the class to attention for a discussion.

“How many different procedures do you think we’ll find from the 12 groups?” I asked. Their predictions ranged from three to nine, with most feeling there would be four or five different methods. Students then read their procedures. We discussed whether each one would work and whether it differed from the others suggested.

Several groups made use of the information “10 oz.,” which was stamped on the bottom of the glass. However, their methods differed.

I pointed out to the class that these methods worked only when you knew the capacity of the containers.

These students used the 10-ounce capacity of the glass to develop procedures for finding the halfway mark.
Four pairs suggested procedures that used beans. Two suggested filling the container and counting the beans into two equal piles, one suggested using a measuring cup to find half the beans, and the other wanted to use a tablespoon to count scoops of beans. The class decided that these procedures fit into two categories, one for counting and one for measuring.

This pair suggested a five-step procedure using beans.

1st fill the cup with beans to the top
2nd dump all the beans and
3rd count them into equal parts
4th put half of the beans in the cup
5th and mark the mark

In all, the students agreed that they had found seven different procedures. Determining the categories for classifying their procedures was a valuable thinking opportunity.

An Afternote

I organized this experience lesson differently when I tried it with other classes. I brought in an assortment of about a dozen containers—plastic glasses of different sizes; Styrofoam and paper drinking cups; and empty jars and bottles of unusual shapes, such as those used for salad dressing and liquid detergent. I labeled them A, B, C, and so on.

First, I had groups decide how many centimeters up from the bottom they thought the halfway mark would be on each container. Then I asked them to write one procedure that could be used to find the halfway marks of all of the containers. After groups reported and discussed their procedures, each group applied two different procedures to each of two containers. They posted the distances up from the bottom to the halfway marks on each container so that students could compare these with their original predictions.

This problem-solving lesson always engages students and provides for lively discussions.
Geoboard Dot Paper

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Roll for $1.00

You need: A partner or small group
One die
Pennies (about 30)
Dimes (about 20)

1. Each person takes a turn rolling one die.
2. On each turn, all players use the number rolled.
3. Each player takes as many pennies OR dimes as the number rolled. A player may not take both pennies and dimes on the same turn.
4. Each player puts pennies in the Pennies column and dimes in the Dimes column.
5. When a player has 10 or more pennies, he or she MUST exchange 10 pennies for a dime. Put the dime in any box in the Dimes column.
6. Players who go over $1.00 are out of the game and must wait for the next round.
7. The game is over after seven rolls. The winner is the player who has the closest to but not more than $1.00.
PLUS PATTERNS
1. Press \(2 + 2 =\)
   - You should have 4 on the display.
   - Now press = again and again and again.
   - What pattern of numbers appears?
2. Press \(3 + 2 = = = = . . .\)
   - What pattern appears?
3. Press \(2 + 3 = = = = . . .\)
   - What pattern appears?
4. Try other numbers. Write about what you discover.

MINUS PATTERNS
1. Press \(20 - 2 = = = . . .\) until you get to zero.
2. Start with other numbers and then press \(- 2 = = = . . .\)
   - What can you discover about numbers that get to zero and those that do not?
3. Try other combinations of “starting” and “minus” numbers—
   - for example,
   - \(15 - 4 = = = . . .\)
   - \(10 - 3 = = = . . .\)
   - \(15 - 3 = = = . . .\)
   - \(11 - 6 = = = . . .\)
   - What can you learn about pairs of numbers that do or do not get to zero?

DIVISION PATTERNS
1. Press \(8 ÷ 4 =\)
   - You should have 2 on the display.
2. Press \(8 ÷ 3 =\)
   - You should get 2.6666666.
3. Try other division examples. Before pressing =, predict
   - whether the answer will be a whole number (like 2, 3, 5, or 11)
   - or a decimal number (like 2.6666666, 1.25, 3.5, or 0.375).
   - Write about what you discover.
Calculators and the Education of Youth

National Council of Teachers of Mathematics Position Statement

Developed by the NCTM's major committees, influenced by the deliberations of the Delegate Assembly, and approved by the Board of Directors, position statements are intended to guide discussion and influence decisions affecting school mathematics. The Council grants permission for them to be copied for distribution and used whenever they can be helpful in promoting better mathematics education.

Calculators are widely used at home and in the workplace. Increased use of calculators in school will ensure that students’ experiences in mathematics will match the realities of every day life, develop their reasoning skills, and promote the understanding and application of mathematics. The National Council of Teachers of Mathematics therefore recommends the integration of the calculator into the school mathematics program at all grade levels in classwork, homework, and evaluation.

Instruction with calculators will extend the understanding of mathematics and will allow all students access to rich, problem-solving experiences. This instruction must develop students’ ability to know how and when to use a calculator. Skill in estimation and the ability to decide if the solution to a problem is reasonable and essential adjuncts to the effective use of the calculator.

Evaluation must be in alignment with normal, everyday use of calculators in the classroom. Testing instruments that measure students’ understanding of mathematics and its applications must include calculator use. As the availability of calculators increases and the technology improves, testing instruments and evaluation practices must be continually upgraded to reflect these changes. The National Council of Teachers of Mathematics recommends that all students use calculators to—

- explore and experiment with mathematical ideas such as patterns, numerical and algebraic properties, and functions;
- develop and reinforce skills such as estimation, computation, graphing, and analyzing data;
- focus on problem-solving processes rather than the computations associated with problems;
- perform the tedious computations that often develop when working with real data in problem situations;
- gain access to mathematical ideas and experiences that go beyond those levels limited by traditional paper-and-pencil computation.

The National Council of Teachers of Mathematics also recommends that every mathematics teacher at every level promote the use of calculators to enhance mathematics instruction by—

- modeling the use of calculators in a variety of situations;
- using calculators in computation, problem solving, concept development, pattern recognition, data analysis, and graphing;
- incorporating the use of calculators in testing mathematical skills and concepts;
- keeping current with the state-of-the-art technology appropriate for the grade level being taught;
- exploring and developing new ways to use calculators to support instruction and assessment.

The National Council of Teachers of Mathematics also recommends that—

- school districts conduct staff development programs that enhance teachers’ understanding of the use of appropriate state-of-the-art calculators in the classroom;
- teacher preparation institutions develop preservice and in-service programs that use a variety of calculators, including graphing calculators, at all levels of the curriculum;
- educators responsible for selecting curriculum materials make choices that reflect and support the use of calculators in the classroom;
- publishers, authors, and test and competition writers integrate the use of calculators at all levels of mathematics;
- mathematics educators inform students, parents, administrators, and school boards about the research that shows the advantages of including calculators as an everyday tool for the student of mathematics.

Research and experience have clearly demonstrated the potential of calculators to enhance students’ learning in mathematics. The cognitive gain in number sense, conceptual development, and visualization can empower and motivate students to engage in true mathematical problem solving at a level previously denied to all but the most talented. The calculator is an essential tool for all students of mathematics.
The Game of Leftovers

You need: A partner
One die
15 Color Tiles
One cup to hold the tiles
Six paper plates or 3-inch paper squares (“plates”)

1. Take turns. On your turn, roll the die, take that number of paper plates or squares, and divide the tiles among them. Keep any leftover tiles.

2. Both players record the math sentence that describes what happened.
   
   For example: \(15 \div 4 = 3 \text{ R3}\)
   
   In front of each sentence write the initial of the person who rolled the die.

3. Return the tiles on the plates to the cup before the next player takes a turn.

4. Play until all the tiles are gone. Then figure your scores by counting how many tiles each of you has. The winner is the player with the most leftovers. Add your scores to make sure that they total the 15 tiles you started with.

5. When you finish a game, look at each of your sentences with a remainder of zero (R0). Write on the class chart each sentence with R0 that isn’t already posted.
10-by-10 Grids
Probability Tile Game Statements

1. All 3 tiles are the same color.
2. All the tiles are red.
3. There is a red tile among them.
4. Not all tiles are the same color.
5. There are 2 red tiles among them.
6. Only 1 tile is red; the other 2 tiles are a different color.
7. There is no blue tile among them.
8. There is a green tile among them.
9. There is 1 blue tile.
2-Centimeter Grid Paper

[20x20 grid with 2-centimeter spacing]
Isometric Dot Paper


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MARILYN BURNS

MARILYN BURNS is a nationally known mathematics educator whose messages about teaching math have reached teachers through her many Math Solutions Professional Development books, videotapes, audiotapes, and extensive inservice programs. She is the recipient of numerous awards, including the Glenn Gilbert National Leadership Award, given by the National Council of Supervisors of Mathematics, and the Louise Hay Award for Contributions to Mathematics Education, given by the Association for Women in Mathematics.

50 PROBLEM-SOLVING LESSONS

Since 1986, Marilyn Burns has been writing and publishing the Math Solutions newsletter to offer support to teachers searching for new ways to teach mathematics. Each issue, which now can be found on our Web site at www.mathsolutions.com, is filled with articles grounded in the realities of the classroom. The articles present new ideas for classroom teaching, share new approaches to existing ideas, offer tips for classroom organization, and address general issues about math education.

In addition, each issue contains classroom activities, presented as vignettes, aimed at grades 1–6. Many lessons come from Marilyn Burns’s colleagues or from correspondence with teachers across the country who have attended Math Solutions Inservice courses. 50 Problem-Solving Lessons is a compilation of the best of these classroom-tested lessons. The lessons span the strands of the math curriculum and are illustrated with actual children’s work.

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