PROOF INVOLVING SETS and INDEXED FAMILIES OF SETS

By:
Ahmad Wachidul Kohar, Sri Rejeki
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2.1 Proof Involving Sets
2.1.1 Terms involving Sets

1. Definition 1
Suppose A and B are sets. \( A \subseteq B \) means “if \( x \in A \), then \( x \in B \)”, that is
\[
(A \subseteq B) \iff (x \in A \rightarrow x \in B) \iff (\forall x \in A)(x \in B)
\]

2. Definition 2
If \( A = B \) are sets, we say that \( (A \subseteq B) \) and \( (B \subseteq A) \), that is
\[
A = B \iff (A \subseteq B) \land (B \subseteq A) \iff (x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A).
\]

3. Definition 3
If A and B are sets, we define the sets
\[
A \cup B = \{ x : x \in A \lor x \in B \}
\]
\[
A \cap B = \{ x : x \in A \land x \in B \}
\]
That is, \( x \in A \cup B \iff (x \in A \lor x \in B) \)
\( x \in A \cap B \iff (x \in A \land x \in B) \)

4. Definition 4
If \( A \cap B = \emptyset \), then A and B are said to be disjoint, that is
\[
A \cap B = \emptyset \iff \neg(\exists x \in A \cap B)
\]
\[
\iff \neg(\exists x \in A)(x \in B)
\]
\[
\iff (\forall x \in A)(x \notin B)
\]
\[
\iff x \in A \rightarrow x \notin B
\]

5. Definition 5
If A and B are sets, we define the difference of A and B as
\[
A \setminus B = \{ x : x \in A \text{ and } x \notin B \} = A \cap B^c
\]

6. Definition 6
If A and B are sets, we define the symmetric difference of A and B as
\[
A \Delta B = \{ x : x \in A \cap B^c \text{ or } x \in A^c \cap B \} = (A \cap B^c) \cup (A^c \cap B)
\]
2.1.2 Direct Proof

At the foundational level, the task in writing the proof of a theorem is to show that $p \rightarrow q$ is a tautology.

**Theorem 1**

If $A \subseteq B$, then $A \cap B = A$

Proof by using direct proof method given as follows:

Suppose $A \subseteq B$, and pick $x \in A \cap B$. Then $x \in A$, so that $A \cap B \subseteq A$. Now pick $x \in A$. Then since $A \subseteq B$, it is also true that $x \in B$, so that $x \in A \cap B$. Thus $A \subseteq A \cap B$, so that $A \cap B = A$.

**Theorem 2**

(DeMorgan's Law). If $A$ and $B$ are sets, then

$$(A \cap B) = A \cup B^c$$

Proof:

Proof: We show that $(A \cap B)^c \subseteq A^c \cup B^c$ and $(A \cap B)^c \supseteq A^c \cup B^c$.

$(\subseteq)$: Pick $x \in (A \cap B)^c$. Then $x \notin A \cap B$, so that either $x \notin A$ or $x \notin B$. We consider each case.

(Case $x \notin A$): If $x \notin A$, then $x \in A^c$. Thus $x \in A^c \cup B^c$.

(Case $x \notin B$): If $x \notin B$, then $x \in B^c$. Thus $x \in A^c \cup B^c$. In either case, we have $(A \cap B)^c \subseteq A^c \cup B^c$.

2.1.3 Proof by contrapositive

The idea of this method is that $p \rightarrow q$ is equivalent with $\neg q \rightarrow \neg p$, meaning that we can use this latter form to prove a statement.

**Example**

If $A \subseteq B$, then $A \setminus B = \emptyset$

Proof: By contrapositive, suppose $A \setminus B \neq \emptyset$. Then there exist $x \in (A \setminus B)$. Thus, $x \in A \cap B^c$, so that $x \in A$ and $x \in B^c$, which means there exist $x \in A$ and $x \in B^c$ or $A \varsubsetneq B$. 
2.1.4 Proof by contradiction

The idea of contradiction method is by showing \( \neg(p \rightarrow q) \) is a contradiction of the statement \( p \rightarrow q \), that is a tautology. Another way to write \( \neg(p \rightarrow q) \) is using its equivalence, which is \( p \land \neg q \).

Example:

Given A and B are sets satisfying \( A \subseteq B \). Prove that \( B^c \subseteq A^c \)

Proof:

By contradiction, we obtain

Suppose \( B^c \subseteq A^c \), then

\[ \exists x \in B^c \land x \in A \]
\[ \exists x \in B^c \land x \in B \text{ (given)} \]
\[ \exists x \in (B^c \cap B) \]
\[ \exists x \in \emptyset \text{ (Contradict with the given that is } x \in B) \]

Exercises 2.1

1. Construct the negation of each statement below.
   a. \( x \in A \cup B \)
   b. \( A \) and \( B \) are disjoint
   c. \( A = B \)
   d. \( x \in A \setminus B \)
   e. \( x \in A \Delta B \)
   f. \( A \subseteq B \cup C \)
   g. \( A \cup B \subseteq C \cap D \)
   h. If \( A \cup C \subseteq B \cup C \), then \( A \subseteq B \)

2. Prove the following:
   a. If \( A \subseteq B \), then \( A \cup B = B \)
   b. If \( A \subseteq B \) and \( B \subseteq C \), then \( A \subseteq C \)
   c. If \( A \subseteq B \), then \( A^c \supseteq B^c \)
   d. If \( A \cap B = \emptyset \), then \( A \Delta B = A \cup B \)

3. Suppose A, B, and C are sets. Consider the following statements.
   \[ A = B \iff A \cup C = B \cup C \ldots \ldots \text{1)} \]
   \[ A = B \iff A \cap C = B \cap C \ldots \ldots \text{2)} \]
For each of equation above, one direction of the implication $\iff$ is true and one is false. Prove the direction that is true, and provide a counterexample for the direction that is false.

4. Prove that $\cap$ distributes over $\cup$ and vice versa.

5. If $X$ and $Y$ are disjoint sets, we sometimes write $X \cup Y$. So, if someone makes a statement like

$$A \cup B = A \cup (B \setminus A)$$

What he is really saying is the compound statement

$$A \cup B = A \cup (B \setminus A) \text{ and } A \cap (B \setminus A) = \emptyset$$

Prove equations (1) by showing both parts of equation (2)

**Proof**

1. The negation of the forms a) to h) are given as follows

   (a) $x \in A \cup B$

   $x \in A \cup B$ means that $x \in A \text{ or } x \in B$

   $x \notin A \cup B$ means that $x \notin A \text{ and } x \notin B$

   (b) $A$ and $B$ are disjoint.

   $A$ and $B$ are disjoint means that $A \cap B = \emptyset$

   It means there is no $x \in A$ and $x \in B$

   $x \notin A$ and $x \notin B$

   Thus, the negation $x \notin A \cap x \notin B$ is $x \in (A \cup B)$

   (c) $A = B$

   $A = B \iff (A \subseteq B) \cap (B \subseteq A)$

   $\iff (x \in A \rightarrow x \in B) \cap (x \in B \rightarrow x \in A)$

   $-(A = B) \iff -[(A \subseteq B) \cap (B \subseteq A)]$

   $\iff -(x \in A \rightarrow x \in B) \cap -(x \in B \rightarrow x \in A)$

   $\iff (x \in A \land x \notin B) \cup (x \in B \land x \notin A)$

   $\iff (A \notin B) \cup (B \notin A)$

   $\iff A \neq B$

   (d) $x \in A \setminus B$

   $x \in A \setminus B \iff x \in A \cap B^c$

   $\iff x \in A \cap x \notin B$

   $x \notin A \setminus B \iff \sim (x \in A \cap x \notin B)$
\( x \notin A \cup x \in B \)
\( x \in A^c \cup x \in B \)
\( x \in A^c \cup B \)
\( x \in B \cup A^c \)
\( x \in B \setminus A \)

(e) \( x \in A \setminus B \)
\( x \in A \setminus B \Rightarrow x \in \{(A \cap B^c) \cup (A^c \cap B)\} \)
\( \Leftrightarrow (x \in A \cap B^c) \cup (x \in A^c \cap B) \)
\( x \notin A \setminus B \Leftrightarrow \{x \in (A \cap B^c) \cup (x \in A^c \cap B)\} \)
\( \Leftrightarrow x \notin A \cap B^c \cap x \notin A^c \cap B \)
\( \Leftrightarrow (x \notin A \cup x \notin B) \cap (x \in A \cup x \notin B) \)
\( \Leftrightarrow (x \in B) \cap (x \in A) \)
\( \Leftrightarrow x \in A \cap x \in B \)
\( \Leftrightarrow x \in A \cap B \)

(f) \( A \subseteq B \cup C \)
\( A \subseteq B \cup C \Leftrightarrow x \in A \to (x \in B \cup C) \)
\( \to (A \subseteq B \cup C) \Leftrightarrow x \in A \cap \neg (x \in B \cup C) \)
\( \Leftrightarrow x \in A \cap (x \notin B \cap x \notin C) \)

(g) \( A \cup B \subseteq C \cap D \)
\( A \cup B \subseteq C \cap D \Leftrightarrow x \in A \cup B \to x \in C \cap D \)
\( \Leftrightarrow (\forall x \in A \cup B)(x \in C \cap D) \)
\( \to (A \cup B \subseteq C \cap D) \Leftrightarrow (\exists x \in A \cup B)(x \notin C \cap D) \)
\( \Leftrightarrow (\exists x \in A \cup B)(x \notin C \cap D) \)

(h) If \( A \cup C \subseteq B \cup C \), then \( A \subseteq B \)
\( A \cup C \subseteq B \cup C \to A \subseteq B \Leftrightarrow (\forall x \in (A \cup C \subseteq B \cup C))(x \in A \subseteq B) \)
\( \to (A \cup C \subseteq B \cup C \to A \subseteq B) \Leftrightarrow (\exists x \in (A \cup C \subseteq B \cup C))(x \notin A \subseteq B) \)
\( \Leftrightarrow (\exists x \in (A \cup C \subseteq B \cup C))(x \notin A \subseteq B) \)

2. a. \( A \subseteq B \) means \( (x \in A \to x \in B) \)........a)
\( B \subseteq C \) means \( (x \in B \to x \in C) \)........b)

Using syllogism, from a) and b), we obtain \( (x \in A \to x \in C) \), or \( A \subseteq C \)
b. Since \( A \cap B = \emptyset \), then \( A \cap B^c = A \). Likewise, \( A \cap B = \emptyset \) also causes \( A^c \cap B = B \)

3. For equation 1)

For \((\Rightarrow)\), it will be proven as true by the following proof.
\[
(\subseteq) A \cup C \subseteq B \cup C
\]

\( A \cup C \) means \( x \in A \) or \( x \in C \). Since \( A=B \), then \( x \in A \leftrightarrow x \in B \). Then, we obtain \( x \in B \) or \( x \in C \). Thus, \( x \in B \cup C \)

\[
(\supseteq) A \cup C \supseteq B \cup C
\]

\( B \cup C \) means \( x \in B \) or \( x \in C \). Since \( A=B \), then \( x \in B \leftrightarrow x \in A \). Then, we obtain \( x \in A \) or \( x \in C \). Thus, \( x \in A \cup C \)

For \((\Leftarrow)\), it will be proven as false by showing a counterexample as follows.

Let \( A = \{1, 2, 3\}; B = \{1, 2, 4\}; C = \{2, 3, 4\} \)
\[
A \cup C = \{1, 2, 3, 4\}
\]
\[
B \cup C = \{1, 2, 3, 4\}
\]
\[
A \cup C = B \cup C \, \text{but} \, A \neq B
\]

For equation 2

For \((\Rightarrow)\), it will be proven as true by the following proof.
\[
(\subseteq) A \cap C \subseteq B \cap C
\]

Pick \( x \in A \cap C \), then \( x \in A \) and \( x \in C \)

Since \( x \in A \) and \( A = B \), then \( x \in B \). Since \( x \in B \) and \( x \in C \), so \( x \in B \cap C \). Thus
\[
A \cap C \subseteq B \cap C
\]

\[
(\supseteq) B \cap C \subseteq A \cap C
\]

Pick \( x \in B \cap C \), then \( x \in B \) and \( x \in C \). Since \( x \in B \) and \( A = B \), then \( x \in A \). Because of \( x \in A \) and \( x \in C \), so \( x \in A \cap C \)

Thus \( B \cap C \subseteq A \cap C \)

For \((\Leftarrow)\), it will be proven as false by showing a counterexample as follows.

Let \( A = \{1, 2, 3\}; B = \{2, 3\}; C = \{2\} \)
\[
A \cap C = \{2\}
\]
\[
B \cap C = \{2\}
\]
\[
A \cap C = B \cap C \, \text{but} \, A \neq B
\]

4. a) \( A \cap (B \cup C) = \{ \{x/x \in A \land x \in (B \cup C)\} \)
\[
= \{x/x \in A \land (x \in B \lor x \in C)\} \)
\[\begin{align*}
\text{b)} \quad & A \cup (B \cap C) = \{x/ x \in A \vee x \in (B \cap C)\} \\
& = \{x/ x \in A \vee (x \in B \wedge x \in C)\} \\
& = \{x/ (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\} \\
& = (A \cup B) \cap (A \cup C)
\end{align*}\]

5. \(A \cup B = (A \cup B) \cap U = (A \cup B) \cap (A \cup A^C) = A \cup (B \cap A^C) = A \cup (B \setminus A)\).

a) \(A \cap (B \setminus A) = A \cap (B \cap A^C) = (A \cap B) \cap (A \cap A^C) = (A \cap B) \cap \emptyset = \emptyset \)

Since a) and b) are proven, it is enough to show that \(A \cup B = A \cup (B \setminus A)\)

### 2.2 Indexed Families of Sets

If we're working with a few sets at a time, it's probably sufficient to use \(A, B,\) and \(C\) to represent them. Yet, if we have many sets, for instance, 10 sets (generally called a family or collection of sets instead of a set of sets), it might be more sensible to put them into a family and address them as \(A_1, A_2, \ldots, A_{10}\). In a case like this, we would say that the set \(\{1, 2, 3, \ldots, 10\}\) indexes the family of sets. If we write

\[N_n = \{1, 2, 3, \ldots, n\},\]

then we could write a family of \(n\) sets as

\[A_1, A_2, A_3, \ldots, A_n^k : k \in N_n, \quad A_k^m \subseteq A\]

and we would say that \(N_n\) is an index set for the family of sets. This notation has advantages, for then we could write unions and intersections more succinctly:

\[A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = \bigcup_{k=1}^n A_k\]

\[A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n = \bigcap_{k=1}^n A_k\]

**Definition 2.2.3.**

Let \(\mathcal{F}\) be a family of sets. Then the union over \(\mathcal{F}\) is defined by:

\[\bigcup_{A \in \mathcal{F}} A = \{A : A \in \mathcal{F}, A \subseteq A\}\]

If \(\mathcal{F}\) is indexed by \(A\), this becomes
Definition 2.2.4.
Let $F$ be a family of sets. Then the intersection over $F$ is defined by:

$$\bigcap_{A \in F} A = \bigcap_{\alpha \in A} \bigcap_{A \in A}$$

If $F$ is indexed by $A$, this becomes:

$$\bigcap_{A \in F} A = \bigcap_{\alpha \in A} \bigcap_{A \in A}$$

Example 2.2.5.
Construct the negation of the statement: $x \not\in \bigcup_{F} A$.

Solution:
the statement $x \not\in \bigcup_{F} A$ means $\forall A \cup \bigcap_{A} \notin A$. If $F$ is indexed by $A$, we have

$$\forall \alpha \cup \bigcap_{A} \notin A$$

Example 2.2.5.
Construct the negation of the statement: $x \not\in \cap_{F} A$.

Solution:
the statement $x \not\in \cap_{F} A$ means $\forall A \cap \bigcap_{A} \notin A$. If $F$ is indexed by $A$, we have

$$\forall \alpha \cap \bigcap_{A} \notin A$$

Theorem 2.2.6. (DeMorgan’s Law)
Let $F$ be a family of sets. Then,

$$\left( \bigcup_{F} A \right)' = \bigcap_{F} A$$

Proof:
We prove by mutual subset inclusion

$\subseteq$: Pick $x \in \left( \bigcup_{F} A \right)'$. Then $x \not\in \bigcup_{F} A$. Therefore, for all $A \in F$, $x \not\in A$. But then $x \in A'$ for every $A'$ \in $F$. Thus, $x \in \bigcap_{F} A$.

$\supseteq$: Pick $x \in \bigcap_{F} A$. Then $x \in A'$ for every $A \in F$. Thus, $x \in A$ for every $A \in F$, so that $x \in \bigcup_{F} A$. Therefore, $x \in \left( \bigcup_{F} A \right)'$. 
Theorem 2.2.7. (DeMorgan’s Law)
Let F be a family of sets. Then,
\[ \bigcap_{F} A = \bigcup_{F} A' \]

Proof:
We prove by mutual subset inclusion
\( \subseteq \): Pick \( x \in [\bigcap_{F} A]' \). Then \( x \in \bigcap_{F} A \). Therefore, there exists \( A \in F \), \( x \notin A \). But then \( x \in A' \) for any \( A' \in F \). Thus, \( x \in \bigcup_{F} A \).

\( \supseteq \): Pick \( x \in \bigcup_{F} A' \). Then \( x \in A' \) for any \( A \in F \). Thus, \( x \in A \) for any \( A \in F \), so that \( x \in \bigcap_{F} A \). Therefore, \( x \in [\bigcap_{F} A]' \).

Theorem 2.2.8.
Let \( F \) be a family of sets indexed by \( A \), and suppose \( B \subseteq A \). Then,
1. \( \bigcup_{\beta \in B} A_{\beta} \subseteq \bigcup_{\alpha \in A} A_{\alpha} \).
2. \( \bigcap_{\beta \in B} A_{\beta} \supseteq \bigcap_{\alpha \in A} A_{\alpha} \).

Proof:
1. Pick \( x \in \bigcup_{\beta \in B} A_{\beta} \). Then \( x \in A_{\beta} \) for any \( \beta \in B \). Since \( B \subseteq A \), it follows that \( \beta \in A \). Therefore, \( x \in A_{\alpha} \), for any \( \alpha \in A \). Since \( \beta \) was chosen arbitrarily, we have shown that \( x \in A_{\alpha} \), for any \( \alpha \in A \), so that \( \bigcup_{\alpha \in A} A_{\alpha} \).
2. Pick \( x \in \bigcap_{\alpha \in A} A_{\alpha} \). We must show that \( x \in A_{\beta} \) for all \( \beta \in B \), so pick \( \beta \in B \). Since \( B \subseteq A \), it follows that \( \beta \in A \). Therefore, \( x \in A_{\beta} \), because \( x \in A_{\alpha} \) for any \( \alpha \in A \). Since \( \beta \) was chosen arbitrarily, we have shown that \( x \in A_{\beta} \) for all \( \beta \in B \), so that \( x \in \bigcap_{\beta \in B} A_{\beta} \).

Notice how we showed \( x \in \bigcap_{\beta \in B} A_{\beta} \) by picking an arbitrary \( \beta \in B \) and showing that \( x \in A_{\beta} \). This shows that \( x \in A_{\beta} \) for all \( \beta \in B \), so that \( x \in \bigcap_{\beta \in B} A_{\beta} \). We can write Theorem 2.2.8 in a slightly different form if the family of sets is not indexed. If \( F_1 \) is a family of sets and \( F_2 \subseteq F_1 \), we call \( F_2 \) a subfamily of \( F_1 \). Since \( B \subseteq A \) in Theorem 2.2.8, \( \{ A_{\beta} \}_{\beta \in B} \) is a subfamily of \( \{ A_{\alpha} \}_{\alpha \in A} \). Swapping the notation in Theorem 2.2.8 for an arbitrary family \( F_1 \) and a subfamily \( F_2 \), we have the following.
**Theorem 2.2.9.**

Suppose $\mathcal{F}_1$ is a family of sets, and $\mathcal{F}_2$ is a subfamily of $\mathcal{F}_1$. Then,

1. $\bigcup_{\mathcal{F}_2} A \subseteq \bigcup_{\mathcal{F}_1} A$,

2. $\bigcap_{\mathcal{F}_2} A \supseteq \bigcap_{\mathcal{F}_1} A$. 